Tangent space and the first fundamental form

1. Tangent space

Definition 1. Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface. $p = \mathbf{X}(u_0^1, u_0^2)$ for some (u_0^1, u_0^2) in U. Then the *tangent space $T_p(M)$ of M at p* is the vector space spanned by $\mathbf{X}_1(u_0^1, u_0^2), \mathbf{X}_2(u_0^1, u_0^2)$. Since $\mathbf{X}_1, \mathbf{X}_2$ are linearly independent, $\dim(T_p(M)) = 2$.

Here $\mathbf{X}_1 = \frac{\partial \mathbf{X}}{\partial u^1}$, etc.

Proposition 1. *$T_p(M)$ is well defined. Namely, suppose $\phi : V \to U$ is a diffeomorphism, $V \subset \mathbb{R}^2$ with coordinates (v^1, v^2). Let $\mathbf{Y} = \mathbf{X} \circ \phi$. Then the vector space spanned by $\frac{\partial \mathbf{X}}{\partial u^1}$, $\frac{\partial \mathbf{X}}{\partial u^2}$, and the vector space spanned by $\frac{\partial \mathbf{Y}}{\partial v^1}$, $\frac{\partial \mathbf{Y}}{\partial v^2}$ are the same.*

Proposition 2. *Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch and let $M = \mathbf{X}(U)$. Let $\alpha(t)$ be a smooth curve in $\mathbb{R}^3$ such that $\alpha(t) \in M$ for all $t \in (a, b)$ passing through a point $p = \alpha(t_0)$ say. Then there is $\epsilon > 0$ and there is a unique smooth curve $\beta(t)$ in U with $t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $\alpha(t) = \mathbf{X}(\beta(t))$ in $(t_0 - \epsilon, t_0 + \epsilon)$.*

Sketch of proof. Let α, p as in the proposition and let $(u_0^1, u_0^2) \in U$ with $\mathbf{X}(u_0^1, u_0^2)$. By the lemma, we may assume that near p, the surface is a graph over xy-plane. Namely, there are open sets $\mathbf{u}_0 \in V \subset U$ and W and a diffeomorphism $\phi : W \to V$ with $\phi^{-1}(\mathbf{u}_0) = (x_0, y_0) \in W$ such that $\mathbf{Y}(x, y) = \mathbf{X} \circ \phi(x, y) = (x, y, f(x, y)$. Now $\alpha(t) \in \mathbf{X}(U)$ so $\alpha(t) = (x(t), y(t), f(x(t), y(t))) = \mathbf{Y}(x(t), y(t))$. Let $\beta(t) = \phi(x(t), y(t))$. Then $\mathbf{X}(\beta(t)) = \alpha(t)$.

□

Corollary 1. *Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface. Then $T_p(M)$ consists of the tangent vectors of smooth curves on M passing through p.*

Definition 2. Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch and let $M = \mathbf{X}(U)$. A nonzero vector N at a point $p = \mathbf{X}(u^1, u^2) \in M$ is called *a normal vector* of M at p if it is orthogonal to $T_p(M)$. A normal vector N at p is called *a unit normal vector* if N has unit length.

Questions: How many normal vectors at a point are there? How many unit normal vectors?

2. First fundamental form

Definition 3. Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface.L The *first fundamental form g* of M at p is the inner product at each $T_p(M)$ given by $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$. The first fundamental form of M is the inner product given by $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$ on every $T_p(M)$ for with $p \in M$.

Sometimes $g(\mathbf{v}, \mathbf{w})$ is written as $\mathrm{I}(\mathbf{v}, \mathbf{w})$.

Let $\mathbf{X} : U \to V \subset M$ be a coordinate parametrization. The *coefficients of the first fundamental form g* with respect to the parametrization are defined as:

$$\begin{cases} E = & g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_u, \mathbf{X}_u \rangle; \\ F = & g(\mathbf{X}_u, \mathbf{X}_v) = \langle \mathbf{X}_u, \mathbf{X}_v \rangle; \\ G = & g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_v, \mathbf{X}_v \rangle. \end{cases}$$

if (u, v) denotes points in U.

If we use (u^1, u^2) instead of (u, v) and let $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$, then we also denote coefficients of the first fundamental form g as

$$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle.$$

3. Length of a curve

Suppose $\alpha(t) = (x(t), y(t), z(t))$ is a smooth curve on M, $a \le t \le b$ such that $\alpha(t) = \mathbf{X}((u(t), v(t))$ in local coordinates. Then the length of α is given by

$$\begin{aligned} \ell &= \int_a^b |\alpha'|(t)dt \\ &= \int_a^b \left(E(\alpha(t))(\frac{du}{dt})^2 + 2F(\alpha(t))\frac{du}{dt}\frac{dv}{dt} + G(\alpha(t))(\frac{dv}{dt})^2 \right)^{\frac{1}{2}} dt \\ &= \int_a^b \left((E(u')^2 + 2Fu'v' + G(v')^2\right)^{\frac{1}{2}} dt. \end{aligned}$$

If we use (u^1, u^2) instead of (u, v) and $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$,

$$\ell = \int_a^b \left(\sum_{i,j=1}^2 g_{ij} \frac{du^i}{dt}\frac{du^j}{dt} \right)^{\frac{1}{2}} dt.$$

So sometimes, the first fundamental form is written symbolically as $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, or $g = \sum_{i,j=1}^2 g_{ij} du^i du^j$.

4. Area of a region

Let $\mathbf{X} : U \to M$ be a parametrization of a regular surface. Let R be a closed and bounded region in $\mathbf{X}(U)$. Let $V = \mathbf{X}^{-1}(R)$. The area of R is given by

$$A(R) = \iint_V |\mathbf{X}_u \times \mathbf{X}_v| du dv = \iint_V \sqrt{EG - F^2}$$

where E, F, G are the coefficients of the first fundamental form w.r.t. this parametrization. It is well-defined: $A(R)$ is independent of parametrization.

Assignment 3, Due 27/9/2018

(1) Prove that the definition of tangent space is independent of the choice of parametrization.

(2) Consider the stereographic projection by $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{X}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right).$$

Show that this is a regular surface patch with image being the unit sphere $\mathbb{S}^2$ with the north pole $(0,0,1)$ deleted. Find also the coefficients of the first fundamental form.

(3) Consider the sphere parametrized by spherical coordinates:

$$\mathbf{X}(u,v) = (\sin v \cos u, \sin v \sin u, \cos v)$$

with $-\pi < u < \pi, 0 < v < \pi$. Find the length of the curve α given by $u = u_0$ and $a \le v \le b$ with $0 < a < b < \pi$. (That is $\alpha(t) = (\sin t \cos u_0, \sin t \sin u_0, \cos t)$, with $a \le t \le b$.) Let $\beta(t)$ be another curve joining $\alpha(a)$ to $\alpha(b)$ on the surface, i.e. $\beta(t) = \mathbf{X}(u(t), v(t))$, $a \le t \le b$ with $\beta(a) = \alpha(a)$, $\beta(b) = \alpha(b)$. Show that $\ell(\beta) \ge \ell(\alpha)$.