MAT 4030: DIFFERENTIAL GEOMETRY
Midterm examination, Oct. 20, 2020

Show all steps clearly in your working. **NO** point will be given if sufficient justification is not provided.

Answer all EIGHT questions.

(1) Let $\alpha(t) = (at, bt^2, t^3)$ where a, b are nonzero constants.
Find $\frac{\tau}{\kappa}$. Show that it is a constant if $4b^4 = 9a^2$.
(*Hint: You may use the formula for κ, τ for curves with general parametrization:*

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}; \quad \tau = -\frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}.$$

)

Sol:

$$\alpha' = (a, 2bt, 3t^2)$$

$$\alpha'' = (0, 2b, 6t)$$

$$\alpha''' = (0, 0, 6)$$

$$\alpha' \times \alpha'' = (6bt^2, -6at, 2ab).$$

Then

$$\begin{aligned}\frac{\tau}{\kappa} &= -\frac{\langle \alpha' \times \alpha'', \alpha''' \rangle |\alpha'|^3}{|\alpha' \times \alpha''|^3} \\ &= -\frac{12ab \times (a^2 + 4b^2t^2 + 9t^4)^{\frac{3}{2}}}{(36b^2t^4 + 36a^2t^2 + 4a^2b^2)^{\frac{3}{2}}}.\end{aligned}$$

If $4b^4 = 9a^2$, then

$$\begin{aligned}b^2a^{-2}(36b^2t^4 + 36a^2t^2 + 4a^2b^2) &= 81t^4 + 36b^2t^2 + 9a^2 \\ &= 9(9t^4 + 4b^2t^2 + a^2).\end{aligned}$$

So it is a constant.

Note: Conversely, if it is a constant, then

$$a^2 + 4b^2t^2 + 9t^4 = C(9b^2t^4 + 9a^2t^2 + a^2b^2)$$

for some constant C. Then

$$a^2 = Ca^2b^2, 4b^2 = 9Ca^2, 9 = 9Cb^2.$$

So $C = b^{-2}$ and $4b^4 = 9a^2$.

(2) Consider the Möbius strip:

$$\mathbf{X}(\theta, v) = \mathbf{a}(\theta) + v\mathbf{w}(\theta)$$

$-\pi < \theta < \pi, \quad -\frac{1}{2} < v < \frac{1}{2}$, where

$$\begin{cases} \mathbf{a}(\theta) & = (\cos\theta, \sin\theta, 0); \\ \mathbf{w}(\theta) & = (\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta). \end{cases}$$

Prove that if $\mathbf{X}_\theta = \lambda\mathbf{w}$, then $\lambda = 0$. (*Note that* $|\mathbf{w}| = 1$). Are $\mathbf{X}_\theta, \mathbf{X}_v$ linearly independent?

Sol: (a) Since $|\mathbf{w}| = 1$, if $\mathbf{X}_\theta = \lambda\mathbf{w}$, we have

$$\begin{aligned} \lambda &= < \lambda\mathbf{w}, \mathbf{w} > \\ &= < \mathbf{X}_\theta, \mathbf{w} > \\ &= < \mathbf{a}' + v\mathbf{w}', \mathbf{w} > \\ &= < \mathbf{a}', \mathbf{w} > + v < \mathbf{w}', \mathbf{w} > \\ &= 0 \end{aligned}$$

(b) $a\mathbf{X}_\theta + b\mathbf{X}_v = 0 \Leftrightarrow a\mathbf{X}_\theta + b\mathbf{w} = 0$.
If $a = 0$, then $b\mathbf{w} = 0$. Since $\mathbf{w} \neq 0$, we have $b = 0$.
If $a \neq 0$, then

$$\mathbf{X}_\theta = -\frac{b}{a}\mathbf{w}.$$

Then by the result in (a), we have $b = 0$. Thus $a\mathbf{X}_\theta = 0$.
[Next we show that $\mathbf{X}_\theta \neq 0$.
If $\mathbf{X}_\theta = 0$, then
$(-\sin\theta, \cos\theta, 0) + v(\frac{1}{2}\cos\frac{1}{2}\theta\cos\theta - \sin\frac{1}{2}\theta\sin\theta, \frac{1}{2}\cos\frac{1}{2}\theta\sin\theta + \sin\frac{1}{2}\theta\cos\theta, -\frac{1}{2}\sin\frac{1}{2}\theta) = 0$.
Thus

$$v\sin\frac{1}{2}\theta = 0.$$

If $v = 0$, then $(-\sin\theta, \cos\theta, 0) = 0$, this is a contradiction.
If $\sin\frac{1}{2}\theta = 0$, then $\theta = 0$ since $-\pi < \theta < \pi$. So

$$(0, 1, 0) + v(\frac{1}{2}, 0, 0) = 0.$$

This is also a contradiction. So $\mathbf{X}_\theta \neq 0$.]
Thus $a = 0$.
So

$$a\mathbf{X}_\theta + b\mathbf{X}_v = 0 \Leftrightarrow a = b = 0,$$

which means $\{\mathbf{X}_\theta, \mathbf{X}_v\}$ is linearly independent.

(3) Consider the surface given by $\{z = xy\}$. Parametrized the surface by

$$\mathbf{X}(u, v) = (u, v, uv).$$

(a) Find the coefficients of the first and second fundamental form with respect to this parametrization.

(b) Find the matrix of the shape operator $\mathcal{S}$ with respect to $\mathbf{N}$ and with respect to the ordered basis $\mathbf{X}_u, \mathbf{X}_v$.

(c) Find the Gaussian curvature and mean curvature of the surface.

Sol:

$$\mathbf{X}_u = (1, 0, v), \mathbf{X}_v = (0, 1, u)$$

$$\mathbf{X}_{uu} = (0, 0, 0), \mathbf{X}_{uv} = (0, 0, 1), \mathbf{X}_{vv} = (0, 0, 0)$$

So

$$\mathbf{N} = (1 + u^2 + v^2)^{-\frac{1}{2}}(-v, -u, 1).$$

(a)

$$E = 1 + v^2, F = uv, G = 1 + u^2,$$

$$e = 0, f = (1 + u^2 + v^2)^{-\frac{1}{2}}, g = 0.$$

(b) The matrix is

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

So the matrix is

$$(1 + u^2 + v^2)^{-\frac{3}{2}} \begin{pmatrix} -uv & 1 + v^2 \\ 1 + u^2 & -uv \end{pmatrix}$$

$$K = -\frac{1}{[1 + u^2 + v^2]^2},$$

$$H = -\frac{uv}{[1 + u^2 + v^2]^{\frac{3}{2}}}.$$

(4) Let $\alpha(s) = (x(s), y(s))$ be a regular curve in $\mathbb{R}^2$ parametrized by arc length with unit tangent $\mathbf{t} = \alpha'$. Define the normal $\mathbf{n}$ so that $\{\mathbf{t}, \mathbf{n}\}$ are positively oriented. Define the curvature κ to be the value so that $\mathbf{t}' = \kappa \mathbf{n}$.

Find the curvature of a circle with radius r under the following two parametrizations:

(i) $\alpha(s) = (r \cos \frac{s}{r}, r \sin \frac{s}{r})$;

(ii) $\beta(s) = (r \cos(-\frac{s}{r}), r \sin(-\frac{s}{r}))$.

Sol: $\alpha'(s) = (x'(s), y'(s))$.
Since $\{\mathbf{t}, \mathbf{n}\}$ is positively oriented and $\alpha(s)$ is parametrized by arc length, we have

$$\mathbf{n} = (-y'(s), x'(s)).$$

Thus

$$\kappa = <\kappa\mathbf{n}, \mathbf{n}> = <\mathbf{t}', \mathbf{n}>$$

(i) $\alpha(s) = (r\cos\frac{s}{r}, r\sin\frac{s}{r})$. Then

$$\alpha'(s) = (-\sin\frac{s}{r}, \cos\frac{s}{r}).$$

So $\alpha(s)$ is parametrized by arc length, then

$$\begin{aligned}\kappa &= <\mathbf{t}', \mathbf{n}> \\ &= <(-\frac{1}{r}\cos\frac{s}{r}, -\frac{1}{r}\sin\frac{s}{r}), (-\cos\frac{s}{r}, -\sin\frac{s}{r})> \\ &= \frac{1}{r}\end{aligned}$$

(ii) $\beta(s) = (r\cos(-\frac{s}{r}), r\sin(-\frac{s}{r}))$. Then

$$\beta'(s) = (-\sin\frac{s}{r}, -\cos\frac{s}{r}).$$

So $\beta(s)$ is parametrized by arc length, then

$$\begin{aligned}\kappa &= <\mathbf{t}', \mathbf{n}> \\ &= <(-\frac{1}{r}\cos\frac{s}{r}, \frac{1}{r}\sin\frac{s}{r}), (\cos\frac{s}{r}, -\sin\frac{s}{r})> \\ &= -\frac{1}{r}\end{aligned}$$

(5) Let M be a regular surface and $p \in M$. Suppose M touches a sphere $\mathbb{S}^2(R)$ of radius R at p (i.e. the tangent planes coincide at p). Suppose M is contained in the interior of $\mathbb{S}^2(R)$. With the unit normal $\mathbf{N}$ pointing insider the sphere. Show that each principal curvature of M at p is at least $\frac{1}{R}$.

What can you say the mean curvature and Gaussian curvature of M at p?

What can you say if M is in the exterior of $\mathbb{S}^2(R)$?

(*Hint: You may assume that* $p = (0, 0, 0)$, *the tangent plane is the xy-plane and* $\mathbf{N} = (0, 0, 1)$ *at* p.)

Sol: Near p, M is given by the graph of function (with principal curvatures k_1, k_2 with principal direction $(1,0,0), (0,1,0)$):

$$p(x,y) = k_1 x^2 + k_2 y^2 + o(x^2+y^2),$$

and the sphere is given by the graph of the function

$$q(x,y) = \frac{1}{R}(x^2 + k_2 y^2) + o(x^2+y^2).$$

Since M is insider the sphere, in this settings, we have must have $p \geq q$. So

$$(k_1 - \frac{1}{R})x^2 + (k_2 - \frac{1}{R})y^2 + o(x^2+y^2) \geq 0$$

near $(0,0)$. Hence $k_i \geq \frac{1}{R}$.

Mean curvature $H \geq \frac{1}{R}$ and Gaussian curvature $K \geq \frac{1}{R^2}$.

If M is outside the sphere, then w.r.t. the unit normal $\mathbf{N}$ as above, the principal curvatures are $\leq \frac{1}{R}$, the mean curvature $H \leq \frac{1}{R}$. But we cannot say too much about the Gaussian curvature. It may be very large. (Consider a very small sphere outside the sphere and touches the sphere.)

Another proof:

(a) Let $\vec{v} \in T_pM$ be a unit vector. Let $\alpha(s) \subset M$ be a smooth curve parametrized by arc length such that

$$\alpha(0) = p, \alpha^{'}(0) = \vec{v}.$$

Let $q = (0,0,R)$, so by the assumption,

$$f(s) = |\alpha(s) - q|^2$$

has a local maximum at $s = 0$. Then $f^{''} \leq 0$, which is

$$< \alpha^{'}(0), \alpha^{'}(0) > + < \alpha(0) - q, \alpha^{''}(0) > \leq 0$$

$$1 + < -R\vec{N}(p), \alpha^{''}(0) > \leq 0$$

$$< \vec{N}(p), \alpha^{''}(0) > \geq \frac{1}{R}$$

$$< -\frac{d}{ds}\Big|_{s=0} \vec{N}(\alpha(s)), \alpha^{'}(0) > \geq \frac{1}{R}$$

$$< S_p(\vec{v}), \vec{v} > \geq \frac{1}{R}$$

for any unit vector $\vec{v} \in T_pM$.

Let $\{\vec{v}_1, \vec{v}_2\}$ be an orthonormal basis of T_pM and $\{k_1, k_2\}$ be the principle curvatures such that

$$S_p(\vec{v}_1) = k_1\vec{v}_1, S_p(\vec{v}_2) = k_2\vec{v}_2.$$

Then by the previous result

$$k_1 =< S_p(\vec{v}_1), \vec{v}_1 >\geq \frac{1}{R}$$

$$k_2 =< S_p(\vec{v}_2), \vec{v}_2 >\geq \frac{1}{R}.$$

(b) Since by (a),

$$k_1 =< S_p(\vec{v}_1), \vec{v}_1 >\geq \frac{1}{R}$$

$$k_2 =< S_p(\vec{v}_2), \vec{v}_2 >\geq \frac{1}{R}.$$

We have

$$H(p) = \frac{1}{2}(k_1 + k_2) \geq \frac{1}{R}$$

$$K(p) = k_1 k_2 \geq \frac{1}{R^2}$$

(c) If we assume $\vec{N}(p) = (0, 0, 1)$, similar arguments give us

$$k_1 \leq \frac{1}{R}$$

$$k_2 \leq \frac{1}{R}$$

$$H(p) = \frac{1}{2}(k_1 + k_2) \leq \frac{1}{R}.$$

But we cannot say too much about the Gaussian curvature. It may be very large.

(6) Let $\mathbf{X} : U \to M$ be a parametrization of a regular surface M. Consider another regular surface $\widetilde{M}$ given by $\mathbf{Y} = \lambda\mathbf{X}$ where $\lambda > 0$ is a constant. Let D be a disk in U. Let A be the area of $\mathbf{X}(D) \subset M$ and $\widetilde{A}$ be the area of $\mathbf{Y}(D) \subset \widetilde{M}$. What is the relation between $A, \widetilde{A}$? What is the relation between the Gauss maps of $M, \widetilde{M}$ and what is the relation between the Gaussian curvatures at the corresponding points in $M, \widetilde{M}$? (I.e. at the points $\mathbf{X}(u, v), \mathbf{Y}(u, v)$ for the same u, v).

Sol: Let $g_{ij}, \widetilde{g}_{ij}$ be the corresponding first fundamental forms of X and Y. Then

$$\widetilde{g}_{ij} =< Y_i, Y_j >=< \lambda X_i, \lambda X_j >= \lambda^2 g_{ij}$$

$$\widetilde{N} = \frac{Y_1 \times Y_2}{|Y_1 \times Y_2|} = \frac{\lambda X_1 \times \lambda X_2}{|\lambda X_1 \times \lambda X_2|} = N$$

$$\widetilde{h}_{ij} =< \widetilde{N}, Y_{ij} >=< N, \lambda X_{ij} >= \lambda h_{ij}.$$

Thus we have

$$\begin{aligned}\widetilde{A} &= \int_D \sqrt{\det(\widetilde{g}_{ij})} \\ &= \int_D \sqrt{\lambda^4 \det(g_{ij})} \\ &= \lambda^2 \int_D \sqrt{\det(g_{ij})} \\ &= \lambda^2 A.\end{aligned}$$

$$\begin{aligned}\widetilde{K} &= \frac{\det(\widetilde{h}_{ij})}{\det(\widetilde{g}_{ij})} \\ &= \frac{\lambda^2 \det(h_{ij})}{\lambda^4 \det(g_{ij})} \\ &= \frac{1}{\lambda^2} K.\end{aligned}$$

–*E*–*N*–*D*–