Regular surfaces 2: Change of coordinates and smooth structure

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Let $U_1 = \mathbf{X}^{-1}(S)$ and $V_1 = \mathbf{Y}^{-1}(S)$.

Then $\mathbf{Y}^{-1} \circ \mathbf{X} : U_1 \to V_1$ is a diffeomorphism.

Proof

Let $p \in S$. Then there is an open set $S_1 \subset S$ such that S_1 is given by the graph $\{(x, y, z)|(x, y) \in \mathcal{O}, z = f(x, y)\}$.

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Now if $(u, v) \in U_1$ with $\mathbf{X}(u, v) \in S_1$, then

$$\mathbf{X}(u, v) = (x(u, v), y(u, v), f(x(u, v), y(u, v)))$$

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Since $\mathbf{X}_u$ and $\mathbf{X}_v$ are linearly independent, we have $(x_u, y_u), (x_v, y_v)$ are linearly independent (why?). This implies $(u, v) \to (x, y)$ is difffeormphic near $\mathbf{X}^{-1}(p)$.

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Similarly, if $(\xi, \eta) \in V_1$, then $(\xi, \eta) \to (x, y)$ is diffeomorphic near $\mathbf{Y}^{-1}(p)$. Hence $(\xi, \eta) \to (u, v)$ is diffeomorphic.

Smooth structure

Definition

(i) *Let M be regular surface and let $f : M \to \mathbb{R}$ be a function. f is said to be smooth if and only if $f \circ \mathbf{X}$ is smooth for all coordinate chart $\mathbf{X} : U \to M$.*

Main point: The concepts are well-defined.

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(ii) *M_1, M_2 be regular surfaces and let $F : M_1 \to M_2$ be a map. F is said to be smooth if and only if the following is true: For any $p \in M_1$ and any coordinate charts $\mathbf{X}$ of p, $\mathbf{Y}$ of $q = F(p)$, $\mathbf{Y}^{-1} \circ \mathbf{X}$ is smooth whenever it is defined.*

Main point: The concepts are well-defined.

Abstract surfaces: a digression

An abstract surface (**differentiable manifold of dimension two**) is a set M together with a family of one-to-one maps $\mathbf{X}_\alpha : U_\alpha \to M$ of open sets $U_\alpha \subset \mathbb{R}^2$ such that:

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Definition

Let $\mathbf{X} : U \to \mathbb{R}^3$ *be a regular surface patch, and let* $M = \mathbf{X}(U)$*. Let* $p \in M$ *be a point in the surface.* $p = \mathbf{X}(u_0^1, u_0^2)$ *for some* (u_0^1, u_0^2) *in* U*. Then the tangent space* $T_p(M)$ *of* M *at* p *is the vector space spanned by* $\mathbf{X}_1(u_0^1, u_0^2), \mathbf{X}_2(u_0^1, u_0^2)$*. Since* $\mathbf{X}_1, \mathbf{X}_2$ *are linearly independent,* $\dim(T_p(M)) = 2$*.*

Here $\mathbf{X}_1 = \dfrac{\partial \mathbf{X}}{\partial u^1}$, etc.

Tangent space is well-defined

Proposition

$T_p(M)$ is well defined. Namely, suppose $\phi : V \to U$ is a diffeomorphism, $V \subset \mathbb{R}^2$ with coordinates (v^1, v^2). Let $\mathbf{Y} = \mathbf{X} \circ \phi$. Then the vector space spanned by $\frac{\partial \mathbf{X}}{\partial u^1}$, $\frac{\partial \mathbf{X}}{\partial u^2}$, and the vector space spanned by $\frac{\partial \mathbf{Y}}{\partial v^1}$, $\frac{\partial \mathbf{Y}}{\partial v^2}$ are the same.

Tangent space consists of tangent vectors of curves on M

Lemma

Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch and let $M = \mathbf{X}(U)$. Let $\alpha(t)$ be a smooth curve in $\mathbb{R}^3$ such that $\alpha(t) \in M$ for all $t \in (a, b)$ passing through a point $p = \alpha(t_0)$ say. Then there is $\epsilon > 0$ and there is a unique smooth curve $\beta(t)$ in U with $t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $\alpha(t) = \mathbf{X}(\beta(t))$ in $(t_0 - \epsilon, t_0 + \epsilon)$.

Sketch of proof.

Let α, p as in the proposition and let $(u_0^1, u_0^2) \in U$ with $\mathbf{X}(u_0^1, u_0^2)$. By the lemma, we may assume that near p, the surface is a graph over xy-plane. Namely, there are open sets $\mathbf{u}_0 \in V \subset U$ and W and a diffeomorphism $\phi : W \to V$ with $\phi^{-1}(\mathbf{u}_0) = (x_0, y_0) \in W$ such that $\mathbf{Y}(x, y) = \mathbf{X} \circ \phi(x, y) = (x, y, f(x, y)$. Now $\alpha(t) \in \mathbf{X}(U)$ so $\alpha(t) = (x(t), y(t), f(x(t), y(t))) = \mathbf{Y}(x(t), y(t))$. Let $\beta(t) = \phi(x(t), y(t))$. Then $\mathbf{X}(\beta(t)) = \alpha(t)$.

□

Tangent space consists of tangent vectors of curves on M, cont.

Corollary

Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface. Then $T_p(M)$ consists of the tangent vectors of smooth curves on M passing through p.

Normals and unit normals

Definition

Let $\mathbf{X} : U \to \mathbb{R}^3$ *be a regular surface patch and let* $M = \mathbf{X}(U)$*. A nonzero vector* N *at a point* $p = \mathbf{X}(u^1, u^2) \in M$ *is called a normal vector of* M *at* p *if it is orthogonal to* $T_p(M)$*. A normal vector* N *at* p *is called a unit normal vector if* N *has unit length.*

Questions: How many normal vectors at a point are there? How many unit normal vectors?

First fundamental form

Definition

Let $\mathbf{X}: U \to \mathbb{R}^3$ *be a regular surface patch, and let* $M = \mathbf{X}(U)$*. Let* $p \in M$ *be a point in the surface.L The* *first fundamental form* g *of* M *at* p *is the inner product at each* $T_p(M)$ *given by* $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$*. The first fundamental form of* M *is the inner product given by* $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$ *on every* $T_p(M)$ *for with* $p \in M$*.*

Sometimes $g(\mathbf{v}, \mathbf{w})$ is written as $\mathrm{I}(\mathbf{v}, \mathbf{w})$.

Let $\mathbf{X} : U \to V \subset M$ be a coordinate parametrization. The *coefficients of the first fundamental form g* with respect to the parametrization are defined as:

$$\begin{cases} E = & g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_u, \mathbf{X}_u \rangle; \\ F = & g(\mathbf{X}_u, \mathbf{X}_v) = \langle \mathbf{X}_u, \mathbf{X}_v \rangle; \\ G = & g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_v, \mathbf{X}_v \rangle. \end{cases}$$

if (u, v) denotes points in U.
If we use (u^1, u^2) instead of (u, v) and let $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$, then we also denote coefficients of the first fundamental form g as

$$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle.$$

Length of a curve

Suppose $\alpha(t) = (x(t), y(t), z(t))$ is a smooth curve on M, $a \leq t \leq b$ such that $\alpha(t) = \mathbf{X}((u(t), v(t))$ in local coordinates.

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Then the length of α is given by

$$\ell = \int_a^b |\alpha'|(t)dt$$
$$= \int_a^b \left(E(\alpha(t))(\frac{du}{dt})^2 + 2F(\alpha(t))\frac{du}{dt}\frac{dv}{dt} + G(\alpha(t))(\frac{dv}{dt})^2 \right)^{\frac{1}{2}} dt.$$

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If we use (u^1, u^2) instead of (u, v) and $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$,

$$\ell = \int_a^b \left(\sum_{i,j=1}^{2} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt} \right)^{\frac{1}{2}} dt.$$

Length of a curve, cont.

So sometimes, the first fundamental form is written symbolically as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

or

$$g = \sum_{i,j=1}^{2} g_{ij} du^i du^j.$$

Area of a region

Let $\mathbf{X} : U \to M$ be a parametrization of a regular surface. Let R be a closed and bounded region in $\mathbf{X}(U)$. Let $V = \mathbf{X}^{-1}(R)$. The area of R is given by

$$A(R) = \iint_V |\mathbf{X}_u \times \mathbf{X}_v| dudv = \iint_V \sqrt{EG - F^2}$$

where E, F, G are the coefficients of the first fundamental form w.r.t. this parametrization. It is well-defined: $A(R)$ is independent of parametrization.