Regular surfaces 2

1. Change of coordinates

Proposition 1. *(Change of coordinates) Let M be a regular surface and let $\mathbf{X}: U \to M$, $\mathbf{Y}: V \to M$ be two coordinate parametrizations. Let $S = \mathbf{X}(U) \cap \mathbf{Y}(V) \subset M$ and let $U_1 = \mathbf{X}^{-1}(S)$ and $V_1 = \mathbf{Y}^{-1}(S)$. Then $\mathbf{Y}^{-1} \circ \mathbf{X}: U_1 \to V_1$ is a diffeomorphism.*

Proof. (Sketch) Let $p \in S$. Then there is an open set $S_1 \subset S$ such that S_1 is given by the graph $\{(x,y,z)|(x,y) \in \mathcal{O}, z = f(x,y)\}$. Now if $(u,v) \in U_1$ with $\mathbf{X}(u,v) \in S_1$, then

$$\mathbf{X}(u,v) = (x(u,v), y(u,v), f(x(u,v), y(u,v))$$

because $z = f(x,y)$. Then

$$\mathbf{X}_u = (x_u, y_u, f_x x_u + f_y y_u), \mathbf{X}_v = (x_v, y_v, f_x x_v + f_y y_v).$$

Since $\mathbf{X}_u$ and $\mathbf{X}_v$ are linearly independent, we have $(x_u, y_u), (x_v, y_v)$ are linearly independent (why?). This implies $(u,v) \to (x,y)$ is difffeormphic near $\mathbf{X}^{-1}(p)$. Similarly, if $(\xi, \eta) \in V_1$, then $(\xi, \eta) \to (x,y)$ is diffeomorphic near $\mathbf{Y}^{-1}(p)$. Hence $(\xi, \eta) \to (u,v)$ is diffeomorphic.

□

Definition 1. (i) Let M be regular surface and let $f: M \to \mathbb{R}$ be a function. f is said to be smooth if and only if $f \circ \mathbf{X}$ is smooth for all coordinate chart $\mathbf{X}: U \to M$.

(ii) M_1, M_2 be regular surfaces and let $F: M_1 \to M_2$ be a map. F is said to be smooth if and only if the following is true: For any $p \in M_1$ and any coordinate charts $\mathbf{X}$ of p, $\mathbf{Y}$ of $q = F(p)$, $\mathbf{Y}^{-1} \circ \mathbf{X}$ is smooth whenever it is defined.

Note that a parametrized curve on M is defined as a curve $\alpha: I \to \mathbb{R}^3$ such that $\alpha(t) \in M$ for all t.