Regular surfaces 1

Definition

A subset $M \subset \mathbb{R}^3$ *is said to be a regular surface if for any* $p \in M$*, there is an open neighborhood* U *of* p *in* M*, an open set* D *in* $\mathbb{R}^2$ *and a map* $\mathbf{X} : D \to M \cap U$ *such that the following are true:*

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(rs2) $d\mathbf{X}$ *is full rank:* $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ *and* $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$ *are linearly independent, for any* $(u, v) \in D$.

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(rs3) $\mathbf{X}$ *is a homeomorphism from D onto $M \cap U$. (That is:* $\mathbf{X}$ *is bijective,* $\mathbf{X}$ *and* $\mathbf{X}^{-1}$ *are continuous).*

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So a regular surface is a set M in $\mathbb{R}^3$ which can be covered by a family of coordinate charts.

X has full rank

X is a homeomorphism

Example 1: graphs, $z = f(x, y)$

Example 2: spheres, $\{x^2 + y^2 + z^2 = 1\}$

Spherical coordinates

Stereographic projection

Graphs are regular surfaces

Proposition

Let $f : U \to \mathbb{R}$ be a smooth function on an open set $U \subset \mathbb{R}^2$. Then the graph of f defined by the following is a regular surface:

$$\mathrm{graph}(f) = \{(x, y, f(x, y))| \ (x, y) \in U\}.$$

Regular surfaces are graphs locally

Proposition

Let M be regular surface and let $\mathbf{X} : U \to M$ be a coordinate parametrization. Then for any $p = (u_0, v_0) \in U$ there is a open set $V \subset U$ with $p \in V$ such that $\mathbf{X}(V)$ is a graph over an open set in one of the coordinate plane.

Review on inverse function theorem

Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map from an open set U to $\mathbb{R}^n$, $F(\mathbf{x}) = \mathbf{y}(\mathbf{x}) =$ where $\mathbf{x} = (x^1, \dots, x^n)$, $\mathbf{y} = (y^1, \dots, y^m)$. Let $\mathbf{x_0} = (x_0^1, \dots, x_0^n) \in U$. The Jacobian matrix of F at $\mathbf{x}_0$ is the $m \times n$ matrix

$$dF_{\mathbf{x}_0} = \left(\frac{\partial y^i}{\partial x^j}(\mathbf{x}_0) \right).$$

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Theorem

(Inverse Function Theorem) *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Suppose $F(\mathbf{x}_0) = \mathbf{y}_0$ and $dF_{\mathbf{x}_0}$ is nonsingular. Then there exist open sets $U \supset V \ni \mathbf{x}_0$ and $W \ni \mathbf{y}_0$, such that F is a diffeomorphism from V to W. That is to say, $F : V \to W$ is bijective and F^{-1} is also smooth on W.*

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$$F(\mathbf{x}) = A\mathbf{x} + G(\mathbf{x}),$$

$G(\mathbf{x}_1) - G(\mathbf{x}_2) = o(|\mathbf{x}_1 - \mathbf{x}_2|)$ as $\mathbf{x}_1, \mathbf{x}_2 \to \mathbf{O}$.

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Hence for any $\epsilon > 0$, we can find $\delta > 0$ such that if $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{0}, \delta) = \{|\mathbf{x}| < \delta\}$, we have ,

$$|F(\mathbf{x}_1) - F(\mathbf{x}_2)| \geq |A(\mathbf{x}_1 - \mathbf{x}_2)| - \epsilon|\mathbf{x}_1 - \mathbf{x}_2|$$

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From this we conclude that F is one-one in $B(\mathbf{0}, \delta)$ if $\epsilon > 0$ is small enough.

Proof (cont.)

Let $\mathbf{y}_1 \in \mathbb{R}^n$. Want to find $\mathbf{x}$ so that $F(\mathbf{x}) = A\mathbf{x} + G(\mathbf{x}) = y_1$.

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$A\mathbf{x}_{n+1} = \mathbf{y}_1 - G(\mathbf{x}_n)$.
There is $\rho > 0$ such that if $|\mathbf{y}_1| < \rho$, then $\mathbf{x}_n \in B(\mathbf{O}, \frac{1}{2}\delta)$ and
$\mathbf{x}_n \to \mathbf{x} \in B(\mathbf{O}, \delta)$. (Why?)

Idea of proof

Proof:

(Sketch) Let $\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v))$. May assume that at (u_0, v_0)

$$\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \neq 0.$$

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Let $(x_0, y_0) = (x(u_0, v_0), y(u_0, v_0))$. By the inverse function theorem, there is a nbh of U_1 of (u_0, v_0) and W of (x_0, y_0) so that $(u,v) \to (x,y)$ has a smooth inverse .

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Then the image of U_1 under $\mathbf{X}$ is of the form

$$\begin{aligned}(x, y) \to & (u(x, y), v(x, y)) \\ \to & (x(u(x, y)), y(u(x, y)), z(u(x, y), v(x, y))) \\ = & (x, y, f(x, y)).\end{aligned}$$

Regular value of a function

Proposition

Let U be an open set in $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose a is a regular value of f. (That is: if $f(x, y, z) = a$, then $\nabla f(x) \neq \mathbf{O}$.) Then

$$M = \{(x, y, z) \in U | \ f(x) = a\}$$

is a regular surface.

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Consider the map: $F : U \to \mathbb{R}^3$ defined by $F(x, y, z) = (x, y, f(x, y, z))$. Then the Jacobian matrix is invertible at $p = (x_0, y_0, z_0)$.

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Let $F(x_0, y_0, z_0) = (u_0, v_0, t_0) = q$, with $t_0 = a$. Then there exist nbh V of p and W of q so that F has a smooth inverse F^{-1}.

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Let $W_1 = \{(u, v)|(u, v, a) \in W\}$. Then for $(x, y, z) \in V \cap M$, $F(x, y, z) = (x, y, a) = (u, v, g(u, v, a))$ and so this set is the graph of over (u, v).

More examples: Quadratic surfaces

Torus

Surfaces of revolution

Ruled surfaces