Regular surfaces 1

1. Definitions

Definition 1. A subset $M \subset \mathbb{R}^3$ is said to be a *regular surface* if for any $p \in M$, there is an open neighborhood U of p in M, an open set D in $\mathbb{R}^2$ and a map $\mathbf{X} : D \to M \cap U$ such that the following are true:

(rs1) $\mathbf{X}$ is smooth.
(rs2) $d\mathbf{X}$ is full rank: $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ and $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$ are linearly independent, for any $(u, v) \in D$.
(rs3) $\mathbf{X}$ is a homeomorphism from D onto $M \cap U$. (That is: $\mathbf{X}$ is bijective, $\mathbf{X}$ and $\mathbf{X}^{-1}$ are continuous).

Let M be a regular surface, a map $\mathbf{X} : U \to V$ where V is an open set of M, satisfying the above conditions. $\mathbf{X}$ is called a *parametrization*, and V is called a *coordinate chart (patch, neighborhood)*. If $\mathbf{X}(u, v) = p$, then (u, v) are called local coordinates of p. So a regular surface is a set M in $\mathbb{R}^3$ which can be covered by a family of coordinate charts.

2. Examples

- Graphs: Let $M = \{(x, y, z)|\ z = f(x, y), (x, y) \in U \subset \mathbb{R}^2\}$.
- Sphere: $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3|\ x^2 + y^2 + z^2 = 1\}$.

$\mathbb{S}^2$ can be covered by the following family of coordinate charts.

(i) One of them is $\mathbf{X}(x, y) = (x, y, \sqrt{1 - (x^2 + y^2)})$, $(x, y) \in D$ which is the unit disk in $\mathbb{R}^2$.

(ii) (Spherical coordinates) One of them is:

$$\mathbf{X}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

with $\{(\theta, \varphi)|\ 0 < \theta < \pi, 0 < \varphi < 2\pi\}$.

(iii) (Stereographic projection) The unit sphere M is considered as the set $x^2 + y^2 + (z - 1)^2 = 1$.

$$\pi : M \setminus \{(0, 0, 2) = N\} \to \mathbb{R}^2\}$$

so that $N, p, \pi(p)$ are on a straight line. Then $\mathbf{X} : \mathbb{R}^2 \to M \setminus \{N\}$ is a coordinate chart.

$$\mathbf{X}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4}\right).$$

3. Basic properties on coordinate chart

Proposition 1. *Let $f : U \to \mathbb{R}$ be a smooth function on an open set $U \subset \mathbb{R}^2$. Then the graph of f defined by the following is a regular surface:*

$$\mathrm{graph}(f) = \{(x, y, f(x, y))| \ \ (x, y) \in U\}.$$

Proposition 2. *Let M be regular surface and let $\mathbf{X} : U \to M$ be a coordinate parametrization. Then for any $p = (u_0, v_0) \in U$ there is a open set $V \subset U$ with $p \in V$ such that $\mathbf{X}(V)$ is a graph over an open set in one of the coordinate plane.*

Proof. (Sketch) Let $\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v))$. May assume that (u_0, v_0)

$$\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \neq 0.$$

Let $(x_0, y_0) = (x(u_0, v_0), y(u_0, v_0))$. By the inverse function theorem, there is a nbh of U_1 of (u_0, v_0) and W of (x_0, y_0) so that $(u, v) \to (x, y)$ has a smooth inverse . Then the image of U_1 under $\mathbf{X}$ is of the form

$$(x, y) \to (u(x, y), v(x, y)) \to (x(u(x, y)), y(u(x, y)), z(u(x, y), v(x, y))) = (x, y, f(x, y)).$$

□

Proposition 3. *Let U be an open set in $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose a is a regular value of f. (That is: if $f(x, y, z) = a$, then $\nabla f(x) \neq \mathbf{0}$.) Then*

$$M = \{(x, y, z) \in U| \ f(x) = a\}$$

is a regular surface.

Proof. (Sketch) Let $(x_0, y_0, z_0) \in M$. May assume that $f_z \neq 0$ at this point. Consider the map: $F : U \to \mathbb{R}^3$ defined by $F(x, y, z) = (x, y, f(x, y, z))$. Then the Jacobian matrix is invertible at $p = (x_0, y_0, z_0)$. Let $F(x_0, y_0, z_0) = (u_0, v_0, t_0) = q$, with $t_0 = a$. Then there exist nbh V of p and W of q so that F has a smooth inverse F^{-1}. Now

$$F^{-1}(u, v, t) = (x, y, g(u, v, t)).$$

Let $W_1 = \{(u, v)|(u, v, a) \in W\}$. Then for $(x, y, z) \in V \cap M$, $F(x, y, z) = (x, y, a) = (u, v, g(u, v, a))$ and so this set is the graph of over (u, v).

□

4. More examples

- Quadratic surfaces.
- torus: rotating a circle $(y-a)^2+z^2=r^2$ about the z-axis. So

$$z^2+\left(\sqrt{x^2+y^2}-a\right)^2=r^2.$$

5. Review on inverse function theorem

Let $F: U\subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map from an open set U to $\mathbb{R}^n$, $F(\mathbf{x})=\mathbf{y}(\mathbf{x})=$ where $\mathbf{x}=(x^1,\dots,x^n)$, $\mathbf{y}=(y^1,\dots,y^m)$. Let $\mathbf{x_0}=(x_0^1,\dots,x_0^n)\in U$. The Jacobian matrix of F at $\mathbf{x}_0$ is the $m\times n$ matrix

$$dF_{\mathbf{x}_0}=\left(\frac{\partial y^i}{\partial x^j}(\mathbf{x}_0)\right).$$

Theorem 1. (Inverse Function Theorem) *Let $F: U\subset\mathbb{R}^n\to\mathbb{R}^n$ be a smooth map. Suppose $F(\mathbf{x}_0)=\mathbf{y}_0$ and $dF_{\mathbf{x}_0}$ is nonsingular. Then there exist open sets $U\supset V\ni \mathbf{x}_0$ and $W\ni\mathbf{y}_0$, such that F is a diffeomorphism from V to W. That is to say, $F: V\to W$ is bijective and F^{-1} is also smooth on W.*

Proof. (Sketch) May assume that $\mathbf{x}_0=\mathbf{0}=\mathbf{y}_0$. Let $A=dF_{\mathbf{x}_0}$. Then

$$F(\mathbf{x})=A\mathbf{x}+G(\mathbf{x})$$

here $F(\mathbf{x})$ and $\mathbf{x}$ are considered as a column vectors with $G(\mathbf{x}_1)-G(\mathbf{x}_2)=o(|\mathbf{x}_1-\mathbf{x}_2|$ as $\mathbf{x}_1,\mathbf{x}_2\to\mathbf{0}$. Hence for any $\epsilon>0$, we can find $\delta>0$ such that if $\mathbf{x}_1,\mathbf{x}_2\in B(\mathbf{0},\delta)=\{|\mathbf{x}|<\delta$, we have

$$|F(\mathbf{x}_1)-F(\mathbf{x}_2)|\ge|A(\mathbf{x}_1-\mathbf{x}_2)|-\epsilon|\mathbf{x}_1-\mathbf{x}_2|$$

From this we conclude that F is one-one in $B(\mathbf{0},\delta)$. (Why?)

Let $\mathbf{y}_1\in\mathbb{R}^n$. Define

$$\mathbf{x}_0=A^{-1}\mathbf{y}_1.$$

In general, define

$$\mathbf{x}_{n+1}=A^{-1}\left(\mathbf{y}_1-G(\mathbf{x}_n)\right)$$

There is $\rho>0$ such that if $|\mathbf{y}_1|<\rho$, then $\mathbf{x}_n\in B(\mathbf{0},\frac{1}{2}\delta)$ and $\mathbf{x}_n\to\mathbf{x}\in B(\mathbf{0},\delta)$. (Why?) ∎