Normal curvatures

Let M be a regular surface. Assume M is orientable with an orientation $\mathbf{N}$. Let $\alpha(s)$ be a smooth curve on M parametrized by arc length so that α' is the unit tangent. Let $\mathbf{n}(s)$ be the unit vector at $\alpha(s)$ such that $\mathbf{n} \in T_{\alpha(s)}(M)$ and such that $\{\alpha', \mathbf{n}, \mathbf{N}\}$ is positively oriented, i.e. $\mathbf{n} = \mathbf{N} \times \alpha'$.

Lemma

α' is a linear combination of $\mathbf{n}$ and $\mathbf{N}$: $\alpha'' = k_g\mathbf{n} + k_n\mathbf{N}$ for some smooth functions k_n and k_g on $\alpha(s)$.

Proof.

Since $\alpha'' \perp \alpha'$ and so it is a l.c. of $\mathbf{n}, \mathbf{N}$. □

Normal curvatures, cont.

Definition

As in the lemma, $k_n(s)$ is called the normal curvature of α at $\alpha(s)$ and $k_g(s)$ is called the geodesic curvature of α at $\alpha(s)$.

Facts:

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1. k_n and k_g depend on the choice of $\mathbf{N}$.
2. We will see later that k_g is intrinsic: it depends only on the first fundamental form *and* the orientation of the surface.
3. Let k be the curvature of α. Suppose k is not zero. Let N_α be the normal of α (recalled $\alpha'' = kN_\alpha$). Then $k_n = k\langle N_\alpha, \mathbf{N}\rangle = k\cos\theta$ where θ is the angle between N and $\mathbf{N}$. If $k = 0$, then $\alpha'' = 0$ and $k_n = k_g = 0$.

Normal curvatures and second fundamental form

We first discuss normal curvature. The geometric meaning of the second fundamental form is the following:

Proposition

Let M be an orientable regular surface patch with an orientation $\mathbf{N}$*. Let* $\mathbb{II}$ *be the second fundamental form of M (w.r.t.* $\mathbf{N}$*) and let* $p \in M$*. Suppose* $\mathbf{v} \in T_p(M)$ *with unit length and suppose* $\alpha(s)$ *is a smooth curve of M parametrized by arclength with* $\alpha(0) = p$ *and* $\alpha'(0) = \mathbf{v}$*. Then*

$$k_n(0) = \mathbb{II}_p(\mathbf{v}, \mathbf{v})$$

where k_n is the normal curvature of α at $\alpha(0) = p$*.*

Proof: $k_n = \langle \alpha'', \mathbf{N} \rangle = -\langle \alpha', \frac{d}{dt}\mathbf{N} \rangle = \langle \mathbf{v}, \mathbf{S}_p(\mathbf{v}) \rangle$ at $t = 0$. From this the result follows.

Normal curvatures and second fundamental form, cont.

Corollary

With the same notation as in the proposition, we have the following: Let α and β be two regular curves parametrized by arc length passing through p. Suppose α and β are tangent at p. Then the normal curvatures of α and β at p are equal.

Hence in order to find all normal curvatures, we only need to consider the so-called normal section.

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  Now consider the circle on the unit sphere with $z = r$, $0 < r < 1$. The curve is $\alpha(s) = (\rho \cos(\rho^{-1}s), \rho \sin(\rho^{-1}s), r)$ with $r^2 + \rho^2 = 1$. $\alpha'' = (-\rho^{-1}\cos(\rho^{-1}s), -\rho^{-1}\sin(\rho^{-1}s), 0)$. The curvature $k = \rho^{-1}$. $\langle \alpha'', N \rangle = -1 = \rho^{-1}\cos\theta$, where θ is the angle between α'' and $\mathbf{N}$.

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- Consider the curve $(t, 0, t^4)$ on the $x - z$ plane. Rotate this about z-axis we get a surface M. The normal curvature at $p = (0, 0, 0)$ is zero by considering the normal section. Hence $\mathcal{S}_p = 0$.

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- Normal curvatures at a point on the cylinder will change from 1 to 0 if we use $\mathbf{N}$ pointing to the axis.

Basic facts on symmetric bilinear form

Let $(V, \langle \cdot\ ,\ \cdot \rangle)$ be a finite dimensional inner product space and let B be a *symmetric* bilinear form on V.

- Let Q be the corresponding quadratic form, $Q(\mathbf{v}) = B(\mathbf{v}, \mathbf{v})$
- A be the corresponding self-adjoint operator: $\langle A(\mathbf{v}), \mathbf{w}) = B(\mathbf{v}, \mathbf{w})$.

Theorem

Let $(V, \langle \cdot\ ,\ \cdot \rangle)$ be a finite dimensional inner product space of dimension n and let B be a symmetric bilinear form. Then there is an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that B is diagonalized. Namely, $B(\mathbf{v}_i, \mathbf{v}_j) = \lambda_i \delta_{ij}$. $\mathbf{v}_i$ is an eigenvector of A with eigenvalue λ_i: $A(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$. Moreover, if $\mathbf{v} = \sum_{i=1}^n x^i \mathbf{v}_i$, then $Q(\mathbf{v}) = \sum_{i=1}^n \lambda_i (x^i)^2$.

Sketch of proof

We just prove the case that $n = 2$. Let S be the set in V with $||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1$. Then $B(\mathbf{v}, \mathbf{v})$ attains maximum on S at some $\mathbf{v}$. Let $\mathbf{v}_1 \in S$ be such that

$$B(\mathbf{v}_1, \mathbf{v}_1) = \max_{\mathbf{v} \in S} B(\mathbf{v}, \mathbf{v}).$$

Let $\mathbf{v}_2 \in S$ such that $\mathbf{v}_1 \perp \mathbf{v}_2$. It is sufficient to prove that $B(\mathbf{v}_1, \mathbf{v}_2) = 0$. Let $t \in \mathbb{R}$ and let

$$f(t) = \frac{B(\mathbf{v}_1 + t\mathbf{v}_2, \mathbf{v}_1 + t\mathbf{v}_2)}{||\mathbf{v}_1 + t\mathbf{v}_2||^2}.$$

Then $f'(0) = 0$. Hence

$$\begin{aligned} 0 =& 2B(\mathbf{v}_1, \mathbf{v}_2) - 2B(\mathbf{v}_1, \mathbf{v}_1)\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \\ =& 2B(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Proof, cont.

Note that $\lambda_2 = B(\mathbf{v}_2, \mathbf{v}_2) = \min_{\mathbf{v} \in S} B(\mathbf{v}, \mathbf{v})$.
Now $\langle A(\mathbf{v}_1), \mathbf{v}_1 \rangle = B(\mathbf{v}_1, \mathbf{v}_1) = \lambda_1 = \lambda_1 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle$;
$\langle A(\mathbf{v}_1), \mathbf{v}_2 \rangle = B(\mathbf{v}_1, \mathbf{v}_2) = 0 = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$. Hence

$$\langle A(\mathbf{v}_1) - \lambda_1 \mathbf{v}_1, \mathbf{v}_i \rangle = 0$$

for $i = 1, 2$. Hence $A(\mathbf{v}_1) = \lambda_1 \mathbf{v}_1$.
Let $\mathbf{v} = \sum_{i=1}^{2} x^i \mathbf{v}_i$, then

$$\begin{aligned} Q(\mathbf{v}) =& B(\mathbf{v}, \mathbf{v}) \\ =& \sum_{i,j=1}^{2} x^i x^j B(\mathbf{v}_i, \mathbf{v}_j) \\ =& \sum_{i=1}^{n} \lambda_i (x^i)^2. \end{aligned}$$

Principal curvatures

Let M be an orientable regular surface with orientation $\mathbf{N}$.

Definition

Let $\mathbf{e}_1$, $\mathbf{e}_2$ be an orthonormal basis on $T_p(M)$ which diagonalizes $\mathbb{II}_p$ with eigenvalues k_1 and k_2. Then k_1, k_2 are called the principal curvatures of M at p and $\mathbf{e}_1$, $\mathbf{e}_2$ are called the principal directions. Suppose $k_1 \leq k_2$ then all normal curvature k must satisfies $k_1 \leq k \leq k_2$.

Proposition

With the above notations, if $k_1 = k_2 = k$, then every direction is a principal direction and in this case, $\mathcal{S}_p = k\mathbf{id}$. (In this case, the point is said to be umbilical.) Moreover, the Gaussian curvature and the mean curvature are given by $K(p) = k_1 k_2$, and $H(p) = \frac{1}{2}(k_1 + k_2)$.