# Principal curvatures

Let $M$ be an orientable regular surface with orientation $\mathbf{N}$.

**Definition**

*Let $\mathbf{e}_1$, $\mathbf{e}_2$ be an orthonormal basis on $T_p(M)$ which diagonalizes $\mathbb{II}_p$ with eigenvalues $k_1$ and $k_2$. Then $k_1, k_2$ are called the principal curvatures of $M$ at $p$ and $\mathbf{e}_1$, $\mathbf{e}_2$ are called the principal directions. Suppose $k_1 \leq k_2$ then all normal curvature $k$ must satisfies $k_1 \leq k \leq k_2$.*

# Principle curvatures and Gaussian curvature, mean curvature

**Proposition**

*With the above notations, if $k_1 = k_2 = k$, then every direction is a principal direction and in this case, $\mathcal{S}_p = k\,\mathbf{id}$. (In this case, the point is said to be umbilical.) Moreover, the Gaussian curvature and the mean curvature are given by $K(p) = k_1 k_2$, and $H(p) = \frac{1}{2}(k_1 + k_2)$. In particular,*

$$k_1, k_2 = H \pm \sqrt{H^2 - K}.$$

# Local structure of the surface in terms of principal curvatures

**Definition**

*Let $p$ be a point in a regular surface patch. Then it is called*

1. *Elliptic if* $\det(\mathcal{S}_p) > 0$.
2. *Hyperbolic if* $\det(\mathcal{S}_p) < 0$
3. *Parabolic if* $\det(\mathcal{S}_p) = 0$ *but* $\mathcal{S}_p \neq 0$.
4. *Planar if* $\mathcal{S}_p = 0$.

# Local structure of the surface in terms of principal curvatures, cont.

Let $M$ be a regular surface and $p \in M$. Let $\mathbf{e}_1, \mathbf{e}_2$ be the principal directions with principal curvature $k_1, k_2$ with $\mathbf{N} = \mathbf{e}_1 \times \mathbf{e}_2$. We choose the coordinates in $\mathbb{R}^3$ as follows: $p$ is the origin, $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$. $M$ is graph over $xy$-plane near $p$. That is: there is an open set $p \in V$ so that

$$M = \{(x, y, z) | \ z = f(x, y), (x, y) \in U \subset \mathbb{R}^2\}$$

where $U$ being open in $\mathbb{R}^2$.

# Local structure of the surface in terms of principal curvatures, cont.

**Proposition**

*Near $p = (0,0,0)$, the surface is the graph of*

$$f(x,y) = \frac{1}{2}(k_1 x^2 + k_2 y^2) + o(x^2 + y^2).$$

*Hence locally, the regular surface patch is a*

- *elliptic paraboloid if $p$ is elliptic;*
- *hyperbolic paraboloid if $p$ is hyperbolic;*
- *parabolic cylinder if $p$ is parabolic;*
- *planar point if $\mathcal{S}_p = 0$.*

**Proof**: $M$ can be parametrized as $\mathbf{X}(x,y) = (x, y, f(x,y))$ locally. $p = (0,0,0)$ implies that $f(0,0) = 0$. Note that $\mathbf{X}_x = (1, 0, f_x), \mathbf{X}_y = (0, 1, f_y), \mathbf{X}_{xx} = (0, 0, f_{xx}), \mathbf{X}_{xy} = \mathbf{X}_{yx} = (0, 0, f_{xy}), \mathbf{X}_{yy} = (0, 0, f_{yy})$. $\mathbf{N} = (1 + f_x^2 + f_y^2)^{-\frac{1}{2}}(-f_x, -f_y, 1)$. $\mathbf{N} = (0,0,1)$, implies that $f_x(0,0) = 0, f_y(0,0) = 0$, we have

$$f(x,y) = \frac{1}{2}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2) + o(x^2 + y^2).$$

$$\mathcal{S}_p(\mathbf{e}_1) = -\frac{\partial}{\partial x}\mathbf{N} = (f_{xx}, f_{xy}, 0) = k_1\mathbf{e}_1.$$

Similar for $\mathbf{e}_2$. So at $p$, $f_{xx} = k_1, f_{xy} = 0, f_{yy} = k_2$. Hence the result.

# Regular surface where all points are umbilical

**Proposition**

*Let* $\mathbf{X} : U \to \mathbb{R}^3$ *be an orientable regular surface, which is connected. Suppose every point in* $M$ *is umbilical. Then* $M$ *is contained in a plane or in a sphere.*

**Proof**: Let us first consider a coordinate patch, $\mathbf{X}(u, v)$ with $(u, v) \in U$ which is connected. Let $\mathbf{N}$ be a unit normal vector field on $M$ and let $\mathcal{S}$ be the shape operator. Then $\mathcal{S}_p(\mathbf{v}) = \lambda \mathbf{v}$ for any $\mathbf{v} \in T_p(M)$ for some function $\lambda(p)$. We write $\lambda = \lambda(u, v)$. This is a smooth function. Now

$$-\mathbf{N}_u = \mathcal{S}_p(\mathbf{X}_u) = \lambda \mathbf{X}_u.$$

Hence $-\mathbf{N}_{uv} = \lambda_v \mathbf{X}_u + \lambda \mathbf{X}_{uv}$. Similarly, $-\mathbf{N}_{vu} = \lambda_u \mathbf{X}_v + \lambda \mathbf{X}_{vu}$. Hence $\lambda_u = \lambda_v = 0$ everywhere (Why?). So $\lambda$ is constant in this coordinate chart. Hence $\lambda$ is constant on $M$. (Why?).

# Proof, cont.

**Case 1**: $\lambda \equiv 0$. Then $\mathbf{N}_u = \mathbf{N}_v = 0$. So $\mathbf{N} = \mathbf{a}$, which is a constant vector. Then

$$\langle \mathbf{X}(u,v) - \mathbf{X}(u_0, v_0), \mathbf{N} \rangle_u = \langle \mathbf{X}_u, \mathbf{N} \rangle = 0.$$

Similar for derivative w.r.t. $v$. Hence $\langle \mathbf{X}(u,v) - \mathbf{X}(u_0, v_0), \mathbf{N} \rangle \equiv 0$ and $M$ is contained in a plane. (Why?)

**Case 2**: $\lambda$ is a nonzero constant. Then

$$(\mathbf{X} + \frac{1}{\lambda}\mathbf{N})_u = \mathbf{X}_u + \frac{1}{\lambda}\mathbf{N}_u = 0.$$

Similar for derivative w.r.t. $v$. So $\mathbf{X} + \frac{1}{\lambda}\mathbf{N}$ is a constant vector $\mathbf{a}$, say. Then $|\mathbf{X} - \mathbf{a}| = 1/|\lambda|$. So $M$ is contained a the sphere of radius $1/|\lambda|$ with center at $\mathbf{a}$. (Why?)

- Let $M$ be an orientable regular surface and let $\mathbf{N}$ be a unit normal vector field. We also denote the Gauss map by $\mathbf{N}$. That is $\mathbf{N} : M \to \mathbb{S}^2$ which is the unit sphere in $\mathbb{R}^3$.

# Gauss map, Gauss image and Gaussian curvature

- Let $M$ be an orientable regular surface and let $\mathbf{N}$ be a unit normal vector field. We also denote the Gauss map by $\mathbf{N}$. That is $\mathbf{N} : M \to \mathbb{S}^2$ which is the unit sphere in $\mathbb{R}^3$.
- At $q \in \mathbf{N}(p) \in \mathbb{S}^2$, we use the unit normal vector $\mathbf{N}(p)$ and we identify $T_p(M)$ to $T_q(\mathbb{S}^2)$.

- Let $M$ be an orientable regular surface and let $\mathbf{N}$ be a unit normal vector field. We also denote the Gauss map by $\mathbf{N}$. That is $\mathbf{N} : M \to \mathbb{S}^2$ which is the unit sphere in $\mathbb{R}^3$.
- At $q \in \mathbf{N}(p) \in \mathbb{S}^2$, we use the unit normal vector $\mathbf{N}(p)$ and we identify $T_p(M)$ to $T_q(\mathbb{S}^2)$.
- Let $\mathbf{X}(u^1, u^2)$ $((u^1, u^2) \in U \subset \mathbb{R}^2)$ be a parametrization of $M$ with orientation determined by $\mathbf{N}$.

# Gauss map, Gauss image and Gaussian curvature

- Let $M$ be an orientable regular surface and let $\mathbf{N}$ be a unit normal vector field. We also denote the Gauss map by $\mathbf{N}$. That is $\mathbf{N} : M \to \mathbb{S}^2$ which is the unit sphere in $\mathbb{R}^3$.
- At $q \in \mathbf{N}(p) \in \mathbb{S}^2$, we use the unit normal vector $\mathbf{N}(p)$ and we identify $T_p(M)$ to $T_q(\mathbb{S}^2)$.
- Let $\mathbf{X}(u^1, u^2)$ $((u^1, u^2) \in U \subset \mathbb{R}^2)$ be a parametrization of $M$ with orientation determined by $\mathbf{N}$.
- Then $\mathbf{N} : U \to \mathbb{S}^2$, where $\mathbf{N}(u^1, u^2) = \mathbf{N}(\mathbf{X}(u^1, u^2))$. Then $d\mathbf{N} = -\mathcal{S}$. If The Gaussian curvature is nonzero at a point $p$, then $\mathbf{N}$ can be considered as a parametrization of $\mathbb{S}^2$ near $q$.

# Area of Gauss image

**Proposition**

*Let $p \in M$. Suppose $K(p) \neq 0$. Let $B_n$ be a sequence of open sets with $B_n \to p$ in the sense that $\sup_{q \in B_n} |p - q| \to 0$ as $n \to \infty$. Let $A_n$ be the area of $B_n$ and $\widetilde{A}_n$ be the area of the Gauss image $\mathbf{n}(B_n)$ of $B_n$. Then*

$$\lim_{n \to \infty} \frac{\widetilde{A}_n}{A_n} = |K(p)|.$$

# Proof

## Proof.

May assume that $B_n$ is the image of $U_n \subset U$ of the parametrization $\mathbf{X}$, so that $p \leftrightarrow (0,0)$. Then $U_n \to (0,0)$ if $B_n \to p$. So

$$A_n = \iint_{U_n} |\mathbf{X}_1 \times \mathbf{X}_2| du^1 du^2,$$

□

# Proof

**Proof.**

May assume that $B_n$ is the image of $U_n \subset U$ of the parametrization $\mathbf{X}$, so that $p \leftrightarrow (0,0)$. Then $U_n \to (0,0)$ if $B_n \to p$. So

$$A_n = \iint_{U_n} |\mathbf{X}_1 \times \mathbf{X}_2| du^1 du^2,$$

$$\widetilde{A}_n = \iint_{U_n} |\mathbf{N}_1 \times \mathbf{N}_2| du^1 du^2.$$

□

# Proof

**Proof.**

May assume that $B_n$ is the image of $U_n \subset U$ of the parametrization $\mathbf{X}$, so that $p \leftrightarrow (0,0)$. Then $U_n \to (0,0)$ if $B_n \to p$. So

$$A_n = \iint_{U_n} |\mathbf{X}_1 \times \mathbf{X}_2| du^1 du^2,$$

$$\widetilde{A}_n = \iint_{U_n} |\mathbf{N}_1 \times \mathbf{N}_2| du^1 du^2.$$

Now $d\mathbf{N} = -\mathcal{S}$, so
$\mathbf{N}_1 \times \mathbf{N}_2 = \det(-\mathcal{S})\mathbf{X}_1 \times \mathbf{X}_2 = K\mathbf{X}_1 \times \mathbf{X}_2$. Hence

$$\frac{\widetilde{A}_n}{A_n} = \frac{\iint_{U_n} |K||\mathbf{X}_1 \times \mathbf{X}_2| du^1 du^2}{\iint_{U_n} |\mathbf{X}_1 \times \mathbf{X}_2| du^1 du^2} \to |K(p)|.$$

□

# Meaning of $K > 0$, $K < 0$

- Since $K = \det(\mathcal{S}) = \det(-d\mathbf{N})$, $K > 0$ means $\mathbf{N}$ is orientation preserving, and $K < 0$ means orientation reversing.

# Meaning of $K > 0, K < 0$

- Since $K = \det(\mathcal{S}) = \det(-d\mathbf{N})$, $K > 0$ means $\mathbf{N}$ is orientation preserving, and $K < 0$ means orientation reversing.
- Hence
$$\iint_M K dA$$
can be considered as the signed area of the Gauss image.

# Examples

- $M$ is a plane. The Gauss image is a point and the Gaussian curvature is zero. The area of the Gauss image is zero.

# Examples

- $M$ is a plane. The Gauss image is a point and the Gaussian curvature is zero. The area of the Gauss image is zero.
- Let $M$ be the circular cylinder. The Gaussian curvature is zero. That Gauss image is a circle. The area of the Gauss image is zero.

# Examples

- $M$ is a plane. The Gauss image is a point and the Gaussian curvature is zero. The area of the Gauss image is zero.
- Let $M$ be the circular cylinder. The Gaussian curvature is zero. That Gauss image is a circle. The area of the Gauss image is zero.
- Let $M$ be the sphere of radius $R$. The Gaussian curvature if $1/R^2$. The Gauss image is the whole unit sphere. So the area of the Gauss image is $4\pi$.

# Examples

- $M$ is a plane. The Gauss image is a point and the Gaussian curvature is zero. The area of the Gauss image is zero.
- Let $M$ be the circular cylinder. The Gaussian curvature is zero. That Gauss image is a circle. The area of the Gauss image is zero.
- Let $M$ be the sphere of radius $R$. The Gaussian curvature if $1/R^2$. The Gauss image is the whole unit sphere. So the area of the Gauss image is $4\pi$.
- Let $M$ be the torus. Then

$$K = \frac{\cos u}{r(a + r\cos u)}.$$

Hence

$$\iint_M K dA = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos u}{r(a + r\cos u)} \cdot r(a + r\cos u) du dv = 0.$$