

Definition

A regular surface M is said to be *minimal* if the mean curvature of M is identically zero.

For a graph $\mathbf{X}(x, y) = (x, y, f(x, y))$. $\mathbf{X}_x = (1, 0, f_x)$,

$\mathbf{X}_y = (0, 1, f_y)$, $\mathbf{X}_{xx} = (0, 0, f_{xx})$, $\mathbf{X}_{xy} = \mathbf{X}_{yx} = (0, 0, f_{xy})$,

$\mathbf{X}_{yy} = (0, 0, f_{yy})$. $\mathbf{N} = (1 + f_x^2 + f_y^2)^{-\frac{1}{2}}(-f_x, -f_y, 1)$. So

$E = 1 + f_x^2$, $G = 1 + f_y^2$, $F = f_x f_y$;

$e = (1 + f_x^2 + f_y^2)^{-\frac{1}{2}} f_{xx}$, $f = (1 + f_x^2 + f_y^2)^{-\frac{1}{2}} f_{xy}$, $g = (1 + f_x^2 + f_y^2)^{-\frac{1}{2}} f_{yy}$.

Minimal if

$$0 = H = \frac{1}{2} \cdot \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{(1 + f_x^2 + f_y^2)^{\frac{3}{2}}}.$$

Or

$$\operatorname{div}\left(\frac{\nabla f}{1 + |\nabla f|^2}\right) = 0.$$

Minimal surfaces in isothermal coordinates

Defintion: Let $\mathbf{X}(u, v)$ be a local parametrization of a regular surface. \mathbf{X} is said to be **isothermal** if $|\mathbf{X}_u| = |\mathbf{X}_v| = \lambda$, and $\langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0$.

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Proposition

Let $\mathbf{X}(u, v)$ be an isothermal coordinate parametrization of a regular surface M . Let $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$. Then

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = 2\lambda^2 H \mathbf{N}$$

where H is the mean curvature.

Proof.

$$\langle \mathbf{X}_{uu}, \mathbf{X}_u \rangle = \frac{1}{2} \langle \mathbf{X}_u, \mathbf{X}_u \rangle_u = \lambda_u.$$



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Hence

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = \langle \mathbf{X}_{uu} + \mathbf{X}_{vv}, \mathbf{N} \rangle \mathbf{N} = (e + g) \mathbf{N} = 2\lambda^2 H \mathbf{N},$$



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because

$$H = \frac{1}{2} \frac{eG - 2fF + Eg}{EG - F^2} = \frac{1}{2} \frac{e + g}{\lambda^2}.$$



Minimal surfaces and complex variables

Corollary: Suppose $\mathbf{X}(u, v)$ is an isothermal coordinate parametrization of a regular surface M . M is a minimal surface if and only if $\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0$. (That is: each coordinate function is harmonic as a function of u, v .)

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Remark: Let $\mathbf{X}(u, v)$ be a coordinate parametrization of M . Let $\phi_1 = x_u - \sqrt{-1}x_v$, $\phi_2 = y_u - \sqrt{-1}y_v$, $\phi_3 = z_u - \sqrt{-1}z_v$. Then

- (i) \mathbf{X} is isothermal if and only if $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$.
- (ii) M is minimal if and only if ϕ_i are analytic for $i = 1, 2, 3$.

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Then $E = G = \cosh^2 v$, $F = 0$.

$$\mathbf{X}_{uu} = (-\cosh v \cos u, -\cosh v \sin u, 0);$$

$$\mathbf{X}_{vv} = (\cosh v \cos u, \cosh v \sin u, 0).$$

So $\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0$. Catenoid is minimal.

Surfaces of revolution which are minimal

Consider the surface of revolution given by

$$\mathbf{X}(u, v) = (f(v) \cos u, f(v) \sin u, g(v)); (f')^2 + (g')^2 = 1$$

It is minimal if and only if

$$0 = H = \frac{1}{2} \frac{-g' + f(g'f'' - g''f')}{f}.$$

Suppose $g' \neq 0$ somewhere, then v can be expressed as a function of z and $f(v) = \phi(g(v))$. We have $\dot{\phi}$ means derivative w.r.t. z etc.

$$f' = \dot{\phi}g', \quad f'' = \ddot{\phi}(g')^2 + \dot{\phi}g''.$$

So we have

$$0 = -g' + \phi \left(g'(\ddot{\phi}(g')^2 + \dot{\phi}g'') - g''\dot{\phi}g' \right) = -g' + \phi\ddot{\phi}(g')^3$$

Surfaces of revolution which are minimal, cont.

So

$$-1 + \phi\ddot{\phi}(g')^2 = 0.$$

Since $(f')^2 + (g')^2 = 1$, so $(g')^2(1 + \dot{\phi}^2) = 1$, and we have

$$\frac{\phi\ddot{\phi}}{1 + \dot{\phi}^2} = 1.$$

Check, $\phi = a \cosh((z + c)/a)$ are solutions.

Hence $g' \neq 0$ and the surface is part of a catenoid, or $g' \equiv 0$, then the surface is a part of a plane.

First variational formula for area: Minimal surfaces are critical points of the areas functional

Let $\mathbf{X} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a coordinate parametrization of a regular surface M . Let \bar{D} be a compact domain in U and let $Q = \mathbf{X}(\bar{D}) \subset M$. Let $h(u, v)$ be a smooth function on \bar{D} . Let $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$ be the unit normal of the surface. Define:

$$\mathbf{Y}(u, v; t) = \mathbf{X}(u, v) + th(u, v)\mathbf{n}(u, v).$$

Lemma

*There exists $\epsilon > 0$ such that for each fixed t with $|t| < \epsilon$, $\mathbf{Y}(u, v; t)$ represent a parametrized regular surface. ($\mathbf{Y}(u, v; t)$ is called a **normal variation** of \bar{Q} .)*

Let $\mathbf{Y}_u = \mathbf{X}_u + t(h_u \mathbf{n} + h \mathbf{n}_u)$, etc. So

$$\begin{aligned}\mathbf{Y}_u \times \mathbf{Y}_v &= \mathbf{X}_u \times \mathbf{X}_v + t[(h_u \mathbf{n} + h \mathbf{n}_u) \times \mathbf{X}_v + \mathbf{X}_u \times (h_v \mathbf{n} + h \mathbf{n}_v)] \\ &\quad + t^2(h_u \mathbf{n} + h \mathbf{n}_u) \times (h_v \mathbf{n} + h \mathbf{n}_v) \\ &= \mathbf{X}_u \times \mathbf{X}_v + R(u, v, t).\end{aligned}$$

Since $|\mathbf{X}_u \times \mathbf{X}_v| \geq C_1$ for some $C_1 > 0$ on \bar{D} and $|R| \leq \epsilon C_2$ for some $C_2 > 0$ on \bar{D} independent of ϵ . So $\mathbf{Y}_u \times \mathbf{Y}_v \neq \mathbf{0}$ if ϵ is small enough.

First variational formula, cont.

Let $\epsilon > 0$ be as above. Define $A(t)$ to be the area of

$$M(t) = \{\mathbf{Y}(u, v, t) \mid (u, v) \in \overline{D}\}.$$

Theorem (First variation of area)

$$\left. \frac{dA}{dt} \right|_{t=0} = -2 \iint_{\overline{Q}} hH dA$$

where H is the mean curvature of M . Here for any function ϕ on \overline{D} ,

$$\iint_{\overline{Q}} \phi dA := \iint_{\overline{D}} \phi |\mathbf{X}_u \times \mathbf{X}_v| dudv.$$

Proof: Let $E(u, v, t) = \langle \mathbf{Y}_u(u, v, t), \mathbf{Y}_u(u, v, t) \rangle$ etc. Let $E_0(u, v) = E(u, v, 0)$ etc (which are the coefficients of the first fundamental form of \mathbf{X}).

$$\begin{aligned} E(u, v, t) &= E_0(u, v) + 2th(u, v)\langle \mathbf{N}_u, \mathbf{X}_u \rangle + O(t^2) \\ &= E_0(u, v) - 2th(u, v)e(u, v) + O(t^2); \end{aligned}$$

$$\begin{aligned} F(u, v, t) &= F_0(u, v) + 2th(u, v)\langle \mathbf{N}_u, \mathbf{X}_v \rangle + O(t^2) \\ &= F_0(u, v) - 2th(u, v)f(u, v) + O(t^2); \end{aligned}$$

$$\begin{aligned} G(u, v, t) &= G_0(u, v) + 2th(u, v)\langle \mathbf{N}_v, \mathbf{X}_v \rangle + O(t^2) \\ &= G_0(u, v) - 2th(u, v)g(u, v) + O(t^2), \end{aligned}$$

where e, f, g are the coefficients of the second fundamental form of \mathbf{X} . Hence

$$EG - F^2 = E_0G_0 - F_0^2 - 2t(eG_0 - 2fF_0 + gG_0) + O(t^2).$$

First variational formula, cont.

Hence

$$\begin{aligned}A(t) &= \iint_{\bar{D}} \sqrt{(EG - F^2)} dudv \\&= \iint_{\bar{D}} \sqrt{E_0 G_0 - F_0^2} dudv - t \iint_{\bar{D}} h \frac{eG_0 - 2fF_0 + gG_0}{\sqrt{E_0 G_0 - F_0^2}} dudv \\&\quad + O(t^2) \\&= \iint_{\bar{D}} \sqrt{E_0 G_0 - F_0^2} dudv - 2t \iint_{\bar{Q}} h H dA + O(t^2).\end{aligned}$$

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- **Corollary:** $A'(0) = 0$ for all normal variation of \bar{Q} if and only if $H \equiv 0$ on Q . Actually, a regular surface M is minimal if and only if $A'(0) = 0$ for all normal variation of M with *compact support*: i.e. any variation by $f\mathbf{N}$ where f satisfies $\bar{f} \neq 0$ is a compact set in M .

Construction of bump function

To prove the theorem, we need to construct a so-called *bump function*, starting with

$$\phi(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ e^{-\frac{1}{t}}, & \text{if } t > 0. \end{cases}$$

Consider the function:

$$\Phi(t) = \frac{\psi_1(t)}{\psi_1(t) + \psi_2(t)}$$

where

$$\psi_1(t) = \phi(2+t)\phi(2-t), \psi_2(t) = \phi(t-1) + \phi(-1-t).$$

Then $\Phi(t)$ satisfies $\Phi(t) \geq 0$, and

$$\Phi(t) = \begin{cases} 1, & \text{if } |t| \leq 1; \\ 0, & \text{if } |t| \geq 2. \end{cases}$$

A general result

Lemma

Let h be a smooth function defined in a domain $U \subset \mathbb{R}^2$. Suppose

$$\iint_U f h \, du \, dv = 0$$

for all smooth function f with compact support in U , then $h \equiv 0$.

A reference for minimal surfaces: [Osseman, A survey of minimal surfaces](#).

Constant mean curvature surfaces

Let M be an regular surface which is the boundary of a domain. Let \mathbf{N} be a unit normal vector field. Consider the variation given by variational vector field $f\mathbf{N}$: Namely in local coordinate patch:

$$\mathbf{Y}(u, v; t) = \mathbf{X}(u, v) + tf\mathbf{N}(u, v).$$

Or in general $\mathbf{Y} = \mathbf{X} + tf\mathbf{N}$ where \mathbf{X} is the position vector of a point in M .

Variation with constraint

We want to compute the variation of the area under the constraint the the volume is fixed. As before, let $A(t)$ be the area of the surface $\mathbf{Y}(t)$. Then we have

$$A'(0) = -2 \iint_M fHdA.$$

Volume constraint

Let $V(t)$ be the volume contained insider $\mathbf{Y}(t)$. So f must be such that $V'(0) = 0$.

Let $\mathbf{X}(u, v)$ be a local parametrization from $U \rightarrow M \subset \mathbb{R}^3$.

Consider the map

$$\mathbf{F}(u, v, w) = \mathbf{X}(u, v) + w\mathbf{N}(u, v) = (x, y, z).$$

Then the volume between $\mathbf{X}(u, v)$ and $\mathbf{Y}(u, v, t)$ is given by

$$V(t) = \iint_U \left(\int_0^{tf(u,v)} J \, dw \right) \, dudv$$

where

$$\begin{aligned} J &= \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \\ &= \mathbf{F}_u \times \mathbf{F}_v \cdot \mathbf{F}_w \\ &= (\mathbf{X}_u + w\mathbf{N}_u) \times (\mathbf{X}_v + w\mathbf{N}_v) \cdot \mathbf{N} \\ &= \mathbf{X}_u \times \mathbf{X}_v \cdot \mathbf{N} + O(w) \\ &= |\mathbf{X}_u \times \mathbf{X}_v| + O(w). \end{aligned}$$

Hence

$$\begin{aligned} V(t) &= t \iint_U f |\mathbf{X}_u \times \mathbf{X}_v| dudv + O(t^2) \text{ and} \\ V'(0) &= \iint_U f |\mathbf{X}_u \times \mathbf{X}_v| dudv. \end{aligned}$$

Theorem

Let M be as above. Suppose M is a critical point of the area functional under normal variation preserving volume. Then M has constant curvature.

Proof.

From above, we have

$$\iint_M fHdA = 0$$

for all f satisfying $\iint_M fdA = 0$. Hence H must be constant. □

Question: Is it possible that $H = 0$?

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- Roulette of a parabola gives a catenoid.
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- Roulette of a hyperbola is called a nodary and it gives a nodoid.