2020年11月26日 8:16

MATH 4030 Tutorial

① Let $\Sigma \subseteq R^3$ be a closed (orientable) $\boxed{\text{connected}}$ surface without boundary, which is not homeomorphic to a sphere.

Prove that there are points on Σ where the Gauss curvature K is positive, negative, and zero respectively.

Pf: By Gauss-Bonnet Theorem, $\cancel{2}, 0, -2, -4, \dots$

$$\iint_\Sigma K = 2\pi \cdot \chi(\Sigma) \leq 2\pi \cdot 0$$

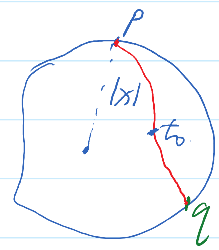
[if we can show that there is a p s.t. $K(p)>0$

then there must be some q s.t. $K(q)<0$]

$\because$ Σ is compact without boundary

$\therefore$ $\exists p \in \Sigma$ s.t.

$K(p) > 0$ [where $|p|$ is a global maximum]



$\therefore$ $\exists\, q \in \Sigma$ s.t.

$K(q) < 0$.

$\therefore$ there is a smooth curve $\alpha \subseteq \Sigma$ s.t.

$\alpha(0) = p$, $\alpha(1) = q$

$\because$ $K(\alpha(t))$ is a smooth function w.r.t. t.

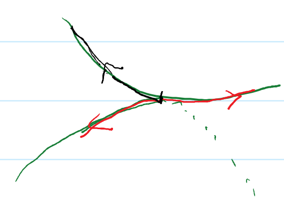
$\left.\begin{array}{l} K(\alpha(0)) = K(p) > 0 \\ K(\alpha(1)) = K(q) < 0 \end{array}\right\} \Rightarrow \exists\, t_0 \in (0,1)$ s.t. $K(\alpha(t_0)) = 0$.

② Let Σ be a surface which is $\boxed{\text{homeomorphic to } R^2}$.

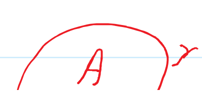
Suppose the Gauss curvature $K<0$ on Σ.

Show that there is no simple closed geodesic.

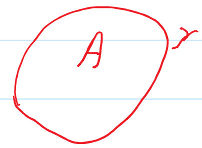
Furthermore, show that there is no closed geodesic.



Pf: (a) Suppose there is a simple closed geodesic γ.



Pf: (a) Suppose there is a simple closed geodesic γ.



$\because$ Σ is homeomorphic to $\mathbb{R}^2$

$\therefore$ γ bounds a region A which is homeomorphic to a disk.

$\therefore$ $\chi(A) = \chi(D) = 1$

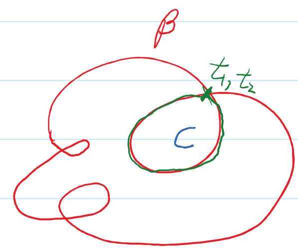
$\therefore$ By the Gauss-Bonnet Theorem,

$$0 > \iint_A K + \int_\gamma \cancel{k_g}^{\,0} = 2\pi \cdot \chi(A) = 2\pi.$$

$\therefore$ $0 > 2\pi$, this is a contradiction.

$\therefore$ such γ can't exist.

(b) Suppose there is a closed geodesic β.

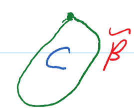


$\therefore$ there is a $t_1 < t_2$ s.t.

$\beta(t_1) = \beta(t_2)$,

$\tilde{\beta} : [t_1, t_2] \to \Sigma$ defined by $\tilde{\beta}(t) = \beta(t)$

is a simple closed curve, and $k_g \equiv 0$ for $t \in (t_1, t_2)$



$\therefore$ By the Gauss-Bonnet Theorem,

$$\iint_C K + \theta_1 = 2\pi \chi(C) = 2\pi.$$

$\therefore$ $\theta_1 = 2\pi - \iint_C K > 2\pi - 0 = 2\pi.$

But $\theta_1 \in [-\pi, \pi]$.

$\therefore$ There is no closed geodesic on Σ.

(Assignment 6, Q4)

Let M be compact surface with positive Gaussian curvature. (orientable, connected, without boundary)

Prove that M is homeomorphic to a sphere.

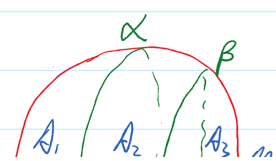
Prove also that if there exist two distinct simple closed geodesics on M, then they must intersect.

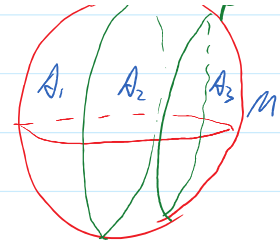
$2, 0, -2, -4, \cdots$

(a) By the Gauss-Bonnet Theorem

$$0 < \iint_M K = 2\pi \cdot \chi(M)$$

$\therefore$ $\chi(M) > 0$





$\therefore \ \chi(M) > 0$

$\therefore \ \chi(M) = 2$

$\therefore$ M is homeomorphic to a sphere.

(b) Suppose we have two distinct simple closed geodesics s.t.

$$\alpha \cap \beta = \phi.$$

$\because$ M is homeomorphic to a sphere

$\therefore$ α, β divide M into three components A_1, A_2, A_3

$\therefore$ A_1, A_3 must be homeomorphic to disks

A_2 . . . . a cylinder.

$\therefore$ By the Gauss-Bonnet Theorem

$$0 < \iint_{A_2} K + \int_\alpha \cancelto{0}{k_g} + \int_\beta \cancelto{0}{k_g} = 2\pi \chi(A_2) = 0.$$

$\therefore \ 0 < 0$

$\therefore \ \alpha \cap \beta \neq \phi.$