MATH 4030 Tutorial

① Prove that a curve $C \subseteq S$ is both an asymptotic curve and a geodesic if and only if C lies on a straight line.

Pf: a curve is asymptotic if

$$\langle \alpha'', N \rangle = 0$$

$$k_n = \langle \alpha'', \vec{N} \rangle = 0$$

$\because \quad \alpha'' = k_g \cdot \vec{n} + k_n \cdot \vec{N}$ where $\{ \frac{\alpha'}{|\alpha'|}, \vec{n}, \vec{N} \}$ is positively oriented

ETS normal direction.

$\therefore$ C is an asymptotic curve and a geodesic

$\Leftrightarrow \quad k_n = 0, \quad k_g = 0$

$\Leftrightarrow \quad \alpha'' = 0$

$\Leftrightarrow \quad \alpha' = \vec{V}$ for some constant vector $\vec{V}$

$\Leftrightarrow \quad \alpha$ lies on some straight line.

② Show that the straight lines are the only geodesics of a plane.

Pf: Let $\alpha(s)$ be any geodesic on a plane.

$\therefore \quad \alpha'' = \cancel{k_g}^{\,0} \cdot \vec{n} + k_n \cdot \vec{N}$

$\quad = k_n \cdot \vec{N}$

$\therefore \quad k_n = \langle \alpha'', \vec{N} \rangle$

$\quad = - \langle \alpha', \frac{d}{ds} \vec{N}(\alpha(s)) \rangle$

$\because \quad \alpha \subseteq$ a plane

$\therefore \quad \vec{N}$ is a constant vector

$\therefore \quad k_n = - \langle \alpha', \frac{d}{ds} \vec{N}(\alpha(s)) \rangle$

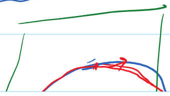
$\quad = 0$

$\therefore \quad \alpha'' = 0$

$\therefore \quad \alpha' = \vec{V}$ for some constant vector $\vec{V}$

$\therefore \quad \alpha$ lies on a straight line.

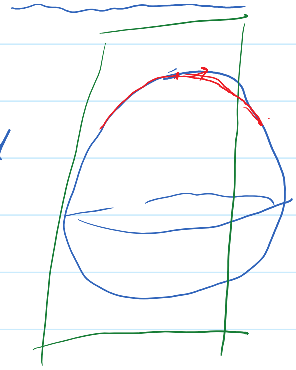
③ Show that if all the geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere.



「 In the previous lectures, if all points of a connected then the surface is contained in a plane or a sphere.

[In the previous lectures, if all points of a connected surface are umbilical points, the the surface is contained in a plane or a sphere.] $\Leftrightarrow$ $S_p = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

$\therefore$ we only need to show that all points are umbilical points.



pf: for any $p \in S$, any unit vector $\vec{v} \in T_pS$

then there is a geodesic $\alpha(s)$ such that

$$\alpha(0) = p, \quad \alpha'(0) = \vec{v}$$

then $\alpha'' = \cancel{k_g}\vec{n} + k_n \cdot \vec{N} = k_n \vec{N} = \langle \alpha'', \vec{N} \rangle \vec{N} = -\langle \alpha', \frac{d}{ds}\vec{N}(\alpha(s)) \rangle \vec{N}$

$\therefore \alpha''(0) = -\langle \vec{v}, S_p(\vec{v}) \rangle \cdot \vec{N}$

$= -(k_1 \cos^2\theta + k_2 \sin^2\theta) \cdot \vec{N}$ where $\vec{v} = \cos\theta v_1 + \sin\theta v_2$

and $\{v_1, v_2\}, \{k_1, k_2\}$ are the corresponding eigenvectors and eigenvalues of the shape operator.

Case 1: if $k_1 = k_2 = 0$

then p is an umbilical point.

Case: if $k_1^2 + k_2^2 \neq 0$

then at least one of k_1, k_2 is not zero.

$\therefore -(k_1\cos^2\theta + k_2 \sin^2\theta)$ has at most four roots in $[0, 2\pi)$.

$\therefore$ for all most all $\theta \in [0, 2\pi)$, $\vec{v} = \cos\theta v_1 + \sin\theta v_2$

we have $\alpha''(0) \neq 0$.

$\because \alpha$ is a geodesic

$\therefore \alpha'' = k_g \vec{n} + k_n \vec{N} = k_n \cdot \vec{N}$.

$\because \alpha'' \neq 0$

we can choose the direction of $\vec{N}$ s.t. $k_n > 0$.

$\therefore k_n(0) = k(0)$ [where k is the curvature of α as a space curve]

and $\vec{N} = \vec{N}_\alpha$ [where $\vec{N}_\alpha$ is the normal vector of α as a space curve $\{T_\alpha, N_\alpha, B_\alpha\}$]

$T_\alpha = \alpha'$
$N_\alpha = \frac{\alpha''}{|\alpha''|}$

$\therefore S_p(\vec{v}) = -\frac{d}{ds}\vec{N}(\alpha(s))\Big|_{s=0} = -\frac{d}{ds}\vec{N}_\alpha(\alpha(s))\Big|_{s=0}$

$= N_\alpha'(\alpha(0)) = -k\,\alpha'(0) + \cancel{\tau} B_\alpha(0)$

$= -k\vec{v}$

$\therefore$ for all most all $\vec{v} \in T_pS$

$S_p(\vec{v}) = -k\vec{v}$

$\therefore p$ is an umbilical point.

$\therefore$ By case 1 and case 2,

all points of S are umbilical points

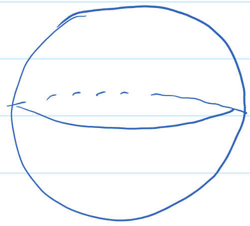
$\therefore$ By the previous lectures,

S lies on a sphere or a plane. #

④ Let $\alpha(s)$ be an arc-length parametrized curve lying on the unit sphere. Let k be the curvature of α as a space curve.

(a) Prove that if $k \geq 1$ everywhere.

(b) if $k \equiv 1$, then α is a geodesic.



$S_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Pf: (a) $\alpha'' = k_g \vec{n} + k_n \vec{N}$

$\therefore k = |\alpha''| = \sqrt{k_g^2 + k_n^2}$

$\because k_n = \langle \alpha'', \vec{N} \rangle = -\langle \alpha', \frac{d}{ds} \vec{N}(\alpha(s)) \rangle$

$= \boxed{\langle \alpha', S_p(\alpha') \rangle}$ α lies on the unit sphere.

$= \boxed{\langle \alpha', \pm\alpha' \rangle = \pm 1}$

$\therefore |k_n| = 1$

$\therefore k = \sqrt{k_g^2 + k_n^2} \geq \sqrt{0+1} = 1$

(b) if $k \equiv 1$, then $k_g \equiv 0$

$\therefore \alpha$ is a geodesic.