2020年11月5日 8:25

MATH 4030 Tutorial

① Does there exist a parametrization $X(u,v): U\subseteq R^2 \to R^3$ of a surface S such that

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad h_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Ans: If there is such surface S.
then the Gauss curvature

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{-1}{1} = -1$$

But, by the Gauss Theorem,

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

$$= 0$$

$\therefore$ it is contradiction.
$\therefore$ there is no such surface with

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad h_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Remark: this means there are some restriction on g_{ij} and h_{ij}
This is so called Gauss-Codazzi eqn.

$$\begin{cases} \partial_k \Gamma_{ij}^l - \partial_j \Gamma_{ik}^l + \Gamma_{ij}^p \Gamma_{pk}^l - \Gamma_{ik}^p \Gamma_{pj}^l = g^{lq}(h_{ij}h_{kq} - h_{ik}h_{jq}) \\ \partial_k h_{ij} - \partial_j h_{ik} + \Gamma_{ij}^p h_{pk} - \Gamma_{ik}^p h_{pj} = 0 \end{cases}$$ for any i,j,k,q.

$$\left[\begin{array}{l} S_p = \cancel{g^{-1}}h \\ S_p = hg^{-1}, \quad H = \frac{1}{2}\operatorname{tr} S_p = \frac{1}{2} g^{ij}h_{ij}. \end{array}\right]$$

② $X: U\subseteq R^2 \to R^3$, Let $S = X(U)$
then the Gauss Theorem tells us

$\{$ K only depends on (g_{ij})

then the Gauss Theorem tells us

$$\begin{cases} K \text{ only depends on } (g_{ij}) \\ \Gamma_{ij}^{k} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\ \ddot{u}^k + \Gamma_{ij}^{k} \dot{u}^i \dot{u}^j = 0. \end{cases}$$

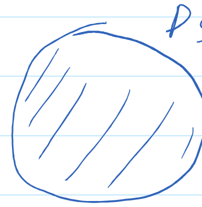
[Abstract Riemannian Surface].

We call (U, g_{ij}) an abstract Riemannian surface.

③ Consider the Poincaré disk model (D, g_{ij}) which is an abstract Riemannian surface with $D = \{(x,y) \in R^2 : x^2+y^2<1\}$ and

$$g_{ij} = \frac{4}{[1-(x^2+y^2)]^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$D \subseteq R^2$



(a) Compute the Christoffel symbols Γ_{ij}^k and the Gauss curvature.

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

$$\Gamma_{11}^{k} = \frac{1}{2} g^{kl}(\partial_1 g_{1l} + \partial_1 g_{1l} - \partial_l g_{11}).$$

$$g_{11} = \frac{4}{[1-(x^2+y^2)]^2}$$

$$= \frac{1}{2} \cdot \frac{[1-(x^2+y^2)]^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left(2 \cdot \begin{bmatrix} \frac{16x}{[1-(x^2+y^2)]^2} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{16x}{[1-(x^2+y^2)]^3} \\ \frac{16y}{[1-(x^2+y^2)]^3} \end{bmatrix}\right)$$

$$\partial_1 g_{11} = \frac{16x}{[1-(x^2+y^2)]^3}.$$

$$= \frac{\cancel{[1-(x^2+y^2)]^2}}{8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{16}{[1-(x^2+y^2)]^{\cancel{3}}}\begin{bmatrix} x \\ -y \end{bmatrix}$$

$$= \frac{2}{[1-(x^2+y^2)]}\begin{bmatrix} x \\ -y \end{bmatrix}.$$

$$\Gamma_{11}^{1} = \frac{2x}{1-(x^2+y^2)}, \qquad \Gamma_{11}^{2} = \frac{-2y}{1-(x^2+y^2)}$$

$$\Gamma_{12}^{k} = \frac{2}{1-(x^2+y^2)}\begin{bmatrix} y \\ x \end{bmatrix}$$

$$\Gamma_{22}^{k} = \frac{2}{1-(x^2+y^2)}\begin{bmatrix} -x \\ y \end{bmatrix}.$$

$K = -e^{-2f}\Delta f$ where $f = \frac{1}{2}\lg E = \frac{1}{2}\lg G = \frac{1}{2}\lg\left(\frac{4}{[1-(x^2+y^2)]^2}\right) = -\lg\frac{1-(x^2+y^2)}{2}$

$$f_x = -\frac{2}{\quad} \cdot \frac{-2x}{\quad} = \frac{2x}{\quad}$$

$$f_x = -\frac{2}{1-(x^2+y^2)}\cdot\frac{-2x}{2} = \frac{2x}{1-(x^2+y^2)}$$

$$\begin{cases} f_{xx} = \dfrac{2[1-(x^2+y^2)]-2x(-2x)}{[1-(x^2+y^2)]^2} = \dfrac{2[1+x^2-y^2]}{[1-(x^2+y^2)]^2} \\ f_{yy} = \dfrac{2[1-x^2+y^2]}{[1-(x^2+y^2)]^2} \end{cases}$$

$$\Delta f = f_{xx}+f_{yy} = \frac{4}{[1-(x^2+y^2)]^2}$$

$$e^{-2f} = e^{-2\cdot\frac{1}{2}\lg G} = \frac{1}{G}.$$

$$K = -e^{-2f}\Delta f = -\frac{[1-(x^2+y^2)]^2}{4}\cdot\frac{4}{[1-(x^2+y^2)]^2} = -1.$$

(b) Compute the geodesic eq^n in this surface.

$$\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j = 0$$

when $k=1$,

$$x'' + \Gamma_{11}^1 (x')^2 + 2\Gamma_{12}^1 x'y' + \Gamma_{22}^1 (y')^2 = 0$$

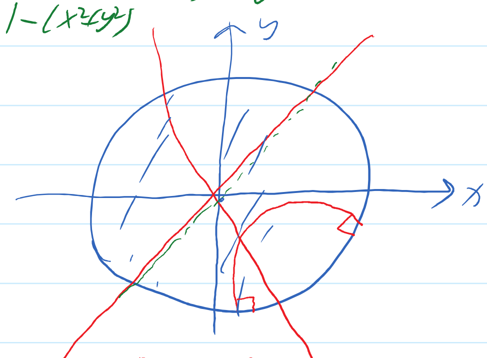
$$\Rightarrow \begin{cases} x'' + \dfrac{2x}{1-(x^2+y^2)}(x')^2 + \dfrac{4y}{1-(x^2+y^2)}x'y' + \dfrac{-2x}{1-(x^2+y^2)}(y')^2 = 0. \\ y'' + \dfrac{2y}{1-(x^2+y^2)}(y')^2 + \dfrac{4x}{1-(x^2+y^2)}x'y' + \dfrac{-2y}{1-(x^2+y^2)}(x')^2 = 0. \end{cases}$$

1°. $(x(t),y(t)) = (at,bt)$

$$\begin{cases} \dfrac{1}{1-(x^2+y^2)}[0 + 2at\cdot a^2 + 4bt\cdot ab - 2at\cdot b^2] \\ \dfrac{1}{1-(x^2+y^2)}[0 + 2bt\cdot b^2 + 4at\cdot ab - 2bt\cdot a^2]. \end{cases}$$

$$(x'',y'') = \frac{1}{1-(x^2+y^2)}\cdot 2(a^2+b^2)(x'(t),y'(t))$$

$\therefore (x(t),y(t))$ is a pre-geodesic.



$$\begin{bmatrix} 2a^3+4ab^2-2ab^2 \\ 2b^3+4a^2b-2a^2b \end{bmatrix} t$$

$$= \begin{bmatrix} (2a^2+2b^2)at \\ (2b^2+2a^2)bt \end{bmatrix}.$$

$$= (2a^2+2b^2)\cdot(x'(t),y'(t))$$

2°