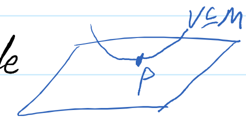
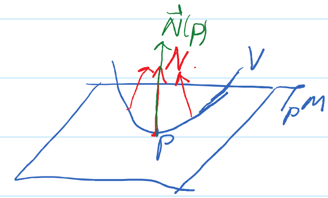
MATH 4030 Tutorial

① A surface M is called locally convex at a point $P\in S$ if there exists a neighborhood $V\subseteq M$ of P such that V lies on one side of T_pM.



If in addition, V has only one intersection point with T_pM, then M is called strictly locally convex at p.

(a) Prove that if M is locally convex at p, then the Gauss curvature $K(p)\geq 0$



Pf: (a) ∵ M is locally convex at p.

∴ By the definition, we choose a unit normal vector field $\vec{N}$ on V such that

$\vec{N}(p)$ points to V.

for any smooth curve $\alpha(s)\subseteq V$ such that

α is p.b.a.l and $\alpha(0)=p$.

∴ define $f(s)=\langle \alpha(s)-p, \vec{N}(p)\rangle$

∵ $\vec{N}(p)$ points to V

∴ $f(s)\geq 0$, $f(0)=\langle p-p, \vec{N}(p)\rangle=0$

∴ $f(s)$ has a local minimum at $s=0$

$$\begin{cases} f'(0)=0 & \Rightarrow \langle \alpha'(0), \vec{N}(p)\rangle=0 \\ f''(0)\geq 0 & \Rightarrow \langle \alpha''(0), \vec{N}(p)\rangle\geq 0 \end{cases}$$

$$\Rightarrow \langle \alpha''(0), \vec{N}(\alpha(0))\rangle\geq 0$$

$$\Rightarrow \langle \alpha'(0), -\frac{d}{ds}\Big|_{s=0}\vec{N}(\alpha(s))\rangle\geq 0$$

∴ Let $\{\vec{v}_1,\vec{v}_2\}$ be an o.n.b of T_pM, and k_1,k_2 be the corresponding principle curvatures, i.e.

$$S_p(\vec{v}_1)=k_1\vec{v}_1,\quad S_p(\vec{v}_2)=k_2\vec{v}_2$$

∵ $\langle \alpha'(0), -\frac{d}{ds}\Big|_{s=0}\vec{N}(\alpha(s))\rangle\geq 0$

$\langle \alpha'(0), S_p(\alpha'(0))\rangle\geq 0$ for any $\alpha'(0)\in T_pM$

∴ $\langle \vec{v}_1, S_p(\vec{v}_1)\rangle\geq 0$

$\langle \vec{v}_1, k_1\vec{v}_1\rangle\geq 0$

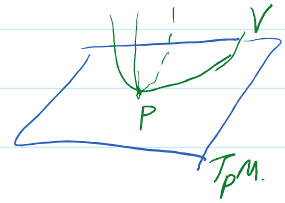
$k_1\geq 0$

Similarly, $k_2 \geq 0$

$\therefore \quad K(p) = k_1 \cdot k_2 \geq 0$.

(b) Prove that if $K(p) > 0$

then M is strictly locally convex at p.



Pf: Let $\vec{N}$ be a unit normal vector field of M.

let $\alpha(s) \subseteq M$ with $\alpha(0) = p$.

then define $f(s) = \langle \alpha(s) - p, \vec{N}(p) \rangle$

$$\begin{cases} f'(0) = \langle \alpha'(0), \vec{N}(p) \rangle = 0 \quad (\alpha'(0) \in T_pM) \\ f''(0) = \langle \alpha''(0), \vec{N}(p) \rangle \\ \quad = \langle \alpha''(0), \vec{N}(\alpha(0)) \rangle \\ \quad = \langle \alpha'(0), -\frac{d}{ds}\Big|_{s=0} \vec{N}(\alpha(s)) \rangle \\ \quad = \langle \alpha'(0), S_p(\alpha'(0)) \rangle \end{cases}$$

Let $\{\vec{v}_1, \vec{v}_2\}$ be an o.n.b of T_pM and k_1, k_2 be the corresponding principle curvatures,

$$S_p(\vec{v}_1) = k_1 \vec{v}_1, \quad S_p(\vec{v}_2) = k_2 \vec{v}_2$$

$\therefore$ for any $\alpha(s) \subseteq M$ with

$\alpha(s)$ is p.b.a.l, and $\alpha(0) = p$.

$\therefore \alpha'(0) = \cos\theta \vec{v}_1 + \sin\theta \vec{v}_2$ for some θ

for $f(s) = \langle \alpha(s) - p, \vec{N}(p) \rangle$

$$\begin{cases} f'(0) = 0 \\ f''(0) = \langle \alpha'(0), S_p(\alpha'(0)) \rangle \end{cases}$$

$$= \langle \cos\theta \vec{v}_1 + \sin\theta \vec{v}_2, S_p(\cos\theta \vec{v}_1 + \sin\theta v_2) \rangle$$

$$= \langle \cos\theta \vec{v}_1 + \sin\theta \vec{v}_2, k_1 \cos\theta \vec{v}_1 + k_2 \sin\theta \vec{v}_2 \rangle$$

$$= k_1 \cos^2\theta + k_2 \sin^2\theta.$$

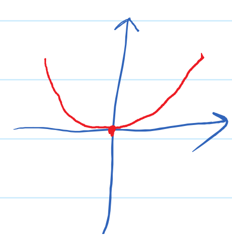
$\because$ By the assumption $K(p) > 0$

$\therefore k_1 \cdot k_2 > 0$

$\therefore$ W.L.O.G, we can assume

$k_1 > 0, \quad k_2 > 0$

$$\begin{cases} f''(0) = k_1 \cos^2\theta + k_2 \sin^2\theta > 0 \\ f'(0) = 0 \\ f(0) = \langle \alpha(0) - p, \vec{N}(p) \rangle = 0 \end{cases}$$



$\quad f(0) = \langle \alpha(0) - p, \vec{N}(p) \rangle = 0$

$\therefore f(s) > 0$ for any s near to 0 and $s \neq 0$.

$\therefore \langle \alpha(s) - p, \vec{N}(p) \rangle > 0$ for any such curve $\alpha(s)$ and $s \neq 0$

$\therefore$ for a neighborhood V of M at p,

$$V \cap T_pM = \{p\}$$

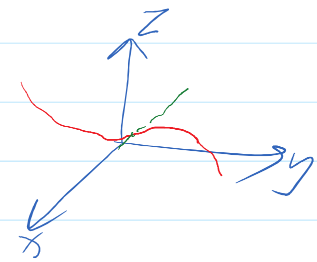
and V lies on one side of T_pM

$\therefore$ M is strictly locally convex at p.

$$\left[\begin{array}{ll} (a): M \text{ is locally convex at } p & \Rightarrow K(p) \geq 0 \\ (b): K(p) > 0 & \Rightarrow M \text{ is strictly locally convex at } p. \end{array}\right.$$

how about $K(p) \geq 0$?

(c) To show that $K(p) \geq 0$ does not imply local convexity, consider $X(x,y) = (x, y, x^3(1+y^2))$ for $y^2 < \frac{1}{2}$.



Ans: for $\boxed{\begin{array}{ll} x<0, & X(x,y)<0 \\ x>0, & X(x,y)>0 \end{array}}$

$$\begin{cases} X_x \big|_{(0,0)} = (1, 0, 3x^2(1+y^2)) \big|_{(0,0)} = (1,0,0) \\ X_y \big|_{(0,0)} = (0, 1, 2y \cdot x^3) \big|_{(0,0)} = (0,1,0) \end{cases}$$

$\therefore$ the surface is not locally convex at $(0,0)$.

But we can compute $K(0,0) \geq 0$

$$\begin{cases} X_x = (1, 0, 3x^2(1+y^2)) \\ X_y = (0, 1, 2yx^3) \end{cases}$$

$$\vec{N}(0,0) = \frac{(1,0,0) \times (0,1,0)}{|(1,0,0) \times (0,1,0)|} = (0,0,1)$$

$$\begin{cases} X_{xx} = (0, 0, 6x(1+y^2)) = (0,0,0) \\ X_{xy} = (0, 0, 6x^2y) = (0,0,0) \\ X_{yy} = (0, 0, 2x^3) = (0,0,0) \end{cases}$$

$$g_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$h_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore K(0,0) = \frac{\det(h)}{\det(g)} = \frac{0}{1} = 0.$$