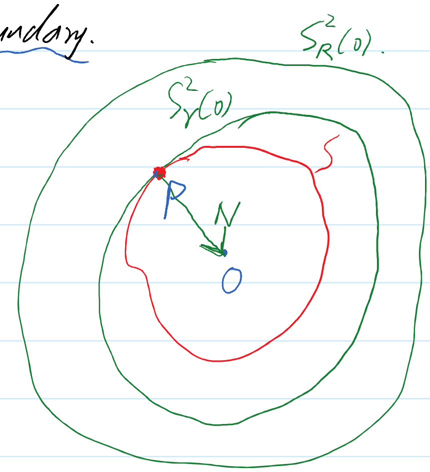
MATH 4030 Tutorial

① Let S be a regular compact surface without boundary. Show that there is a point $P \in S$ such that $K(p) > 0$.



Ans: there is a $r > 0$ such that S is inside $S_r^2(0)$ and $S \cap S_r^2(0)$ at some point P.

1°. Let $\alpha \subseteq S$ be smooth curve p.b.a.l such that $\alpha(0) = p$

$\because$ S is inside $S_r^2(0)$

$\therefore$ $f(s) = \langle \alpha(s), \alpha(s) \rangle$ attains a maximum at the point P.

$\therefore \begin{cases} f'(0) = 0 \\ f''(0) \le 0 \end{cases} \Rightarrow \begin{cases} \langle \alpha'(0), \alpha(0) \rangle = 0 \quad (*) \\ \langle \alpha''(0), \alpha(0) \rangle + \langle \alpha'(0), \alpha'(0) \rangle \le 0 \quad (**) \end{cases}$

$(*) \Rightarrow \alpha(0) \perp \alpha'(0)$

$\because \alpha$ is arbitrary $\therefore$ $\overset{P}{\overset{\|}{\alpha(0)}}$ is a normal vector of S.

$\therefore$ $\vec{N} = \frac{P}{r}$ is a unit normal vector of S at the point P.

$(**)$ $\langle \alpha''(0), \alpha(0) \rangle + 1 \le 0$

$\langle \alpha''(0), \alpha(0) \rangle \le -1$

$\langle \alpha''(0), r \cdot \vec{N}_p \rangle \le -1$

$\therefore \langle \alpha''(0), N_p \rangle \le -\frac{1}{r}$

then $\langle \alpha'(0), -\frac{d}{ds}\big|_{s=0} \vec{N}(\alpha(s)) \rangle = \langle \alpha''(0), N_p \rangle \le -\frac{1}{r}$

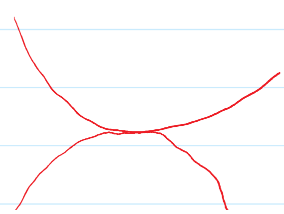
$\langle \alpha'(0), S_p(\alpha'(0)) \rangle \le -\frac{1}{r}$

Let $\{V_1, V_2\}$ be an o.n.b of T_pS and k_1, k_2 be two real numbers such that

$S_p(V_1) = k_1 V_1, \quad S_p(V_2) = k_2 V_2$.

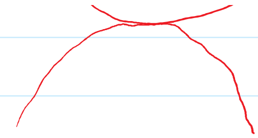
$\Rightarrow \begin{cases} \langle V_1, S_p(V_1) \rangle \le -\frac{1}{r} \\ \langle V_2, S_p(V_2) \rangle \le -\frac{1}{r} \end{cases} \Rightarrow \begin{matrix} k_1 \le -\frac{1}{r} \\ k_2 \le -\frac{1}{r} \end{matrix} \Rightarrow K(p) = k_1 k_2 \ge \frac{1}{r^2} > 0.$ #.

② Let S be a regular surface without boundary.
Suppose $K \le 0$
then S is non-compact.



Pf: Suppose S is compact.

then S is non-compact.



Pf: Suppose S is compact.

$\because$ S is compact without boundary

then by Q1. there is a point $p \in S$ such that

$K(p) > 0$

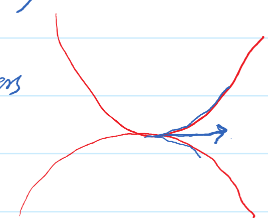
this is a contradiction to the assumption $K \le 0$

$\therefore$ S is non-compact.

In R^n, a subset is compact means it is closed and bounded.

③ Show that if the mean curvature is zero at a non-plannar point, $(k_1^2 + k_2^2 \neq 0)$ then this point has two orthogonal asymptotic direction.

(Def: $\vec{V} \in T_pS$ is called an asymptotic direction if for any smooth curve α p.b.a.l. with $\alpha(0) = p$, $\alpha'(0) = \vec{V}$ we have $k_n(\alpha(0)) = 0$)



Pf: Let $\{V_1, V_2\}$ be an o.n.b of T_pS and k_1, k_2 be two real numbers s.t. $S_p(V_1) = k_1 V_1$, $S_p(V_2) = k_2 V_2$

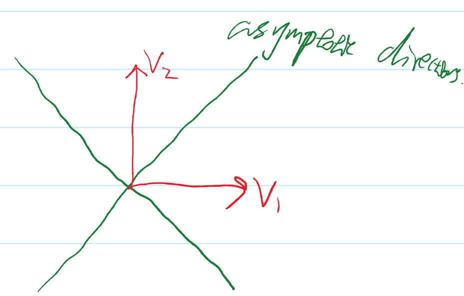
for any unit vector $\vec{V} \in T_pS$, let α be a smooth curve p.b.a.l such that $\alpha(0) = p$, $\alpha'(0) = \vec{V} = \cos\theta \cdot V_1 + \sin\theta V_2$

$$\therefore k_n = \langle \alpha''(s), \vec{N}(\alpha(s)) \rangle$$
$$= \langle \alpha'(s), -\frac{d}{ds}\vec{N}(\alpha(s)) \rangle$$
$$\therefore k_n(\alpha(0)) = \langle \alpha'(0), -\frac{d}{ds}\vec{N}(\alpha(0)) \rangle = \langle \vec{V}, S_p(\vec{V}) \rangle$$
$$= \langle \cos\theta V_1 + \sin\theta V_2, S_p(\cos\theta V_1 + \sin\theta V_2) \rangle$$
$$= \langle \cos\theta V_1 + \sin\theta \cdot V_2, k_1 \cos\theta V_1 + k_2 \sin\theta V_2 \rangle$$
$$= k_1 \cos^2\theta + k_2 \sin^2\theta.$$

$\because H(p) = 0 \Rightarrow k_1 + k_2 = 0$ $(k_1 \neq 0, k_2 \neq 0)$

$$\therefore k_n(\alpha(0)) = k_1 \cos^2\theta - k_1 \sin^2\theta = 0$$

$$\cos^2\theta - \sin^2\theta = 0$$
$$1 - \sin^2\theta - \sin^2\theta = 0$$
$$\sin^2\theta = \frac{1}{2}$$
$$\therefore \sin\theta = \pm\frac{\sqrt{2}}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

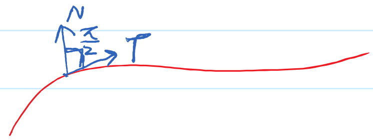


④ (Signed curve for a plane curve).

If $\alpha \subseteq R^3$ is a space curve.

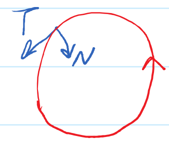
then $N = \frac{T'}{|T'|}$ and $T' = kN \Rightarrow k = |T'| \geq 0$.

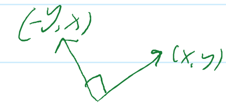
But for a plane curve.



$\{T, N\}_{\mathbb{R}^2}$ positively oriented.

define $k = \langle T', N \rangle$.





- $\alpha(s) = \left(r\cos\frac{s}{r},\ r\sin\frac{s}{r}\right)$

$\therefore \alpha'(s) = \left(-\sin\frac{s}{r},\ \cos\frac{s}{r}\right) \Rightarrow |\alpha'(s)| = 1$

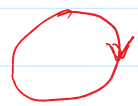
$\therefore T = \left(-\sin\frac{s}{r},\ \cos\frac{s}{r}\right)$

$\therefore N = \left(-\cos\frac{s}{r},\ -\sin\frac{s}{r}\right)$

$\therefore\ k = \langle T', N \rangle = \left\langle \left(-\frac{1}{r}\cos\frac{s}{r},\ -\frac{1}{r}\sin\frac{s}{r}\right),\ \left(-\cos\frac{s}{r},\ -\sin\frac{s}{r}\right) \right\rangle$

$= \frac{1}{r}$

- $\beta = \left(r\cos\left(-\frac{s}{r}\right),\ r\sin\left(-\frac{s}{r}\right)\right)$



$= \left(r\cos\left(\frac{s}{r}\right),\ -r\sin\left(\frac{s}{r}\right)\right)$

$\therefore \beta'(s) = \left(-\sin\frac{s}{r},\ -\cos\frac{s}{r}\right) \Rightarrow |\beta'(s)| = 1$

$\therefore T = \left(-\sin\frac{s}{r},\ -\cos\frac{s}{r}\right)$

$N = \left(\cos\frac{s}{r},\ -\sin\frac{s}{r}\right)$

$k = \langle T', N \rangle = \left\langle \left(-\frac{1}{r}\cos\frac{s}{r},\ \frac{1}{r}\sin\frac{s}{r}\right),\ \left(\cos\frac{s}{r},\ -\sin\frac{s}{r}\right) \right\rangle$

$= -\frac{1}{r}$.