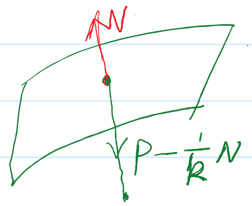
MATH 4030 Tutorial

① Let S be a regular surface with $int(S)$ to be connected.
Let k_1, k_2 be the principle curvatures of S.
Suppose $k_1 = k_2 = k \neq 0$ for some constant k.
Show that S is contained in a sphere with radius $\frac{1}{|k|}$.



Pf: For any $P, q \in S$, $\exists$ a smooth curve p.b.a.l such that $\alpha(0) = P$, $\alpha(l) = q$.

define $C(s) = \alpha(s) + \frac{1}{k}\vec{N}(\alpha(s))$

$|P - C|$ is a constant.

$$\therefore C'(s) = \alpha'(s) + \frac{1}{k}\frac{d}{ds}\vec{N}(\alpha(s))$$
$$= \alpha'(s) - \frac{1}{k} S_{\alpha(s)}(\alpha'(s))$$
$$= \alpha'(s) - \frac{1}{k}\cdot k(\alpha'(s)) = 0.$$

$\therefore C$ is a fixed point.

$$\therefore |\alpha(s) - C| = \left|\frac{1}{k}\vec{N}(\alpha(s))\right| = \frac{1}{k}.$$

$\therefore \alpha$ lies on the sphere with radius $\frac{1}{k}$.

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if $k_1 = k_2 = k$

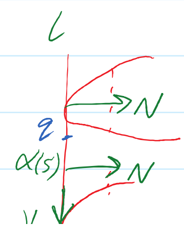
$\therefore \exists \{V_1, V_2\}$ orthonormal basis of T_pS s.t.

$$\begin{cases} S_p(V_1) = k_1 V_1 = kV_1 \\ S_p(V_2) = k_2 V_2 = kV_2 \end{cases} \Rightarrow S_p = k\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\therefore$ for any $\vec{V} = \cos\theta V_1 + \sin\theta V_2$

then $S_p(\vec{V}) = S_p(\cos\theta V_1 + \sin\theta V_2)$ (S_p is linear).
$$= \cos\theta\, S_p(V_1) + \sin\theta\, S_p(V_2)$$
$$= \cos\theta\cdot kV_1 + \sin\theta\, kV_2$$
$$= k(\cos\theta V_1 + \sin\theta V_2) = k\vec{V}.$$

② If a surface S contains a straight line $L \subseteq S$.
Show that $K \leq 0$ at every point $P \in L$.



Pf: [$K = k_1 \cdot k_2$, $K \leq 0 \Leftrightarrow k_1 \leq 0 \leq k_2$].

Step1: [we show that for any $P \in L$, there is a unit vector $V \in T_pS$ s.t. $\langle S_p(V), V \rangle = 0$ (guess $V = \alpha'(s)$)].

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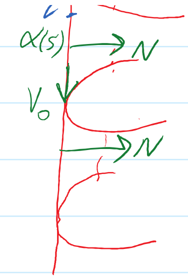


Let $\alpha(s)$ be a parametrization of ℓ p.b.a.l.

$\therefore$ $\alpha(s) = q + s \cdot \vec{V}_0$ for some fixed q, and unit vector $\vec{V}_0$.

$\because$ $\alpha(s) \subseteq \ell \subseteq S$

$\therefore$ $\alpha'(s) \in T_{\alpha(s)}S$.

$\therefore$ $\langle \alpha'(s), \vec{N}(\alpha(s)) \rangle = 0$.

$\therefore$ $\langle \vec{V}_0, \vec{N}(\alpha(s)) \rangle = 0$

$\therefore$ $\langle s \cdot \vec{V}_0, \vec{N} \rangle = 0$

$\therefore$ $\langle \alpha(s) - q, \vec{N} \rangle = 0$.

$\therefore$ $\frac{d}{ds} \langle \alpha(s) - q, \vec{N} \rangle = 0$

$\therefore$ $\langle \alpha'(s), N \rangle + \langle \alpha(s) - q, \frac{d}{ds}\vec{N}(\alpha(s)) \rangle = 0$

$\underbrace{\langle V_0, N \rangle}_{0} + \langle s \cdot V_0, -S_p(V_0) \rangle = 0$. $(\alpha'(s) \equiv V_0)$

$\therefore$ $s \cdot \langle V_0, -S_p(V_0) \rangle = 0$

$\therefore$ $\langle V_0, S_p(V_0) \rangle = 0$ for $s \neq 0$.

$\therefore$ $\langle V_0, S_p(V_0) \rangle = 0$.

$\therefore$ for any $p \in \ell \subseteq S$, $\langle V_0, S_p(V_0) \rangle = 0$.

Step2: let $\{v_1, v_2\}$, $\{k_1, k_2\}$ be the respect eigenvectors and eigenvalues of S_p. s.t.

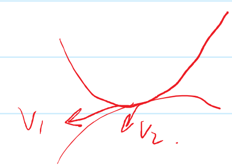
$$\begin{cases} S_p(v_1) = k_1 v_1 \\ S_p(v_2) = k_2 v_2 \\ \{v_1, v_2\} \text{ is a o.n.b.} \end{cases}$$

then $V_0 = \cos\theta\, v_1 + \sin\theta\, v_2$ for some θ

$\therefore$
$$\begin{aligned} 0 &= \langle V_0, S_p(V_0) \rangle \\ &= \langle \cos\theta v_1 + \sin\theta v_2, S_p(\cos\theta v_1 + \sin\theta v_2) \rangle \\ &= \langle \cos\theta v_1 + \sin\theta v_2, \cos\theta \cdot k_1 v_1 + \sin\theta\, k_2 v_2 \rangle. \\ &= \cos^2\theta\, k_1 + \sin^2\theta\, k_2 \end{aligned}$$

[$k_n = \langle \alpha(s), \frac{d}{ds}\vec{N}(\alpha(s)) \rangle$

$\max\{|k_1|, |k_2|\} \leq k_n \leq \max\{|k_1|, |k_2|\}$]



$\therefore$ if $k_1 \leq k_2$

then $k_1 \leq 0 \leq k_2$.

$\therefore$ $K = k_1 \cdot k_2 \leq 0$. #

(3) Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is a constant.

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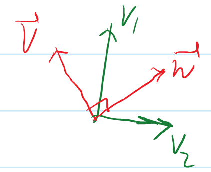
Pf: $\langle \vec{V}, S_p(\vec{V}) \rangle + \langle w, S_p(w) \rangle$ is a constant for any such $\{V, W\}$.



Let $\{V_1, V_2\}$ be an o.n.b of T_pS and $\{k_1, k_2\}$ s.t.

$$S_p(V_1) = k_1 V_1, \quad S_p(V_2) = k_2 V_2.$$

$\because$ $\{V, W\}$ is an o.n.b.

$\therefore$ $\begin{cases} V = \cos\theta V_1 + \sin\theta V_2 \\ W = -\sin\theta V_1 + \cos\theta V_2 \end{cases}$ for some θ.

$$\langle V, S_p(V) \rangle + \langle W, S_p(W) \rangle$$
$$= \langle \cos\theta V_1 + \sin\theta V_2, S_p(\cos\theta V_1 + \sin\theta V_2) \rangle$$
$$+ \langle -\sin\theta V_1 + \cos\theta V_2, S_p(-\sin\theta V_1 + \cos\theta V_2) \rangle$$
$$= \langle \cos\theta V_1 + \sin\theta V_2, \cos\theta k_1 V_1 + \sin\theta k_2 V_2 \rangle$$
$$+ \langle -\sin\theta V_1 + \cos\theta V_2, -\sin\theta k_1 V_1 + \cos\theta k_2 V_2 \rangle$$
$$= k_1 \cos^2\theta + k_2 \sin^2\theta$$
$$+ k_1 \sin^2\theta + k_2 \cos^2\theta$$
$$= k_1 + k_2 = 2H(p).$$

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① $\alpha(t) \subseteq \mathbb{R}^3$ which is not p.b.a.l.

$$k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$
$$\tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}$$

② S.

$$g_{ij} = \langle X_i, X_j \rangle$$
$$h_{ij} = \langle X_{ij}, N \rangle$$
$$S_p = (g_{ij})^{-1} \cdot h_{ij}$$
$$K = \frac{\det(h)}{\det(g)}$$
$$H = \frac{1}{2} g^{ij} \cdot h_{ij} = \frac{1}{2} tr(S_p).$$

$$k_n = \langle \alpha'', N \rangle = \langle \alpha', N \rangle' - \langle \alpha', N' \rangle \qquad \alpha' \in T_pS$$
$$= 0 - \langle \alpha', N' \rangle$$
$$= \langle \alpha', -N' \rangle = \langle \alpha', S_p(\alpha') \rangle$$