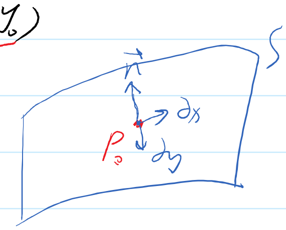
MATH 4030 Tutorial

① Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z=f(x,y)$, at the point $P_0=(x_0, y_0, f(x_0,y_0))$ is given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$



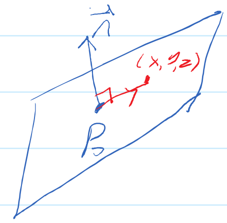
Ans: By assumption, $X: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(x,y) = (x, y, f(x,y))$$

is a parametrization of the surface.

$$X_x = (1, 0, f_x), \qquad X_y = (0, 1, f_y)$$

$$\vec{n} = X_x \times X_y = (-f_x, -f_y, 1).$$



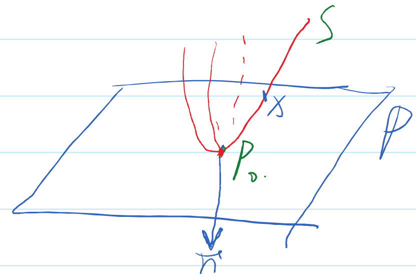
$\therefore$ the equation of the tangent plane is

$$\langle (x,y,z) - (x_0, y_0, f(x_0,y_0)), \vec{n}(P_0) \rangle = 0.$$

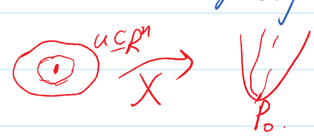
$$\langle (x-x_0, y-y_0, z-f(x_0,y_0)), (-f_x, -f_y, 1) \rangle = 0$$

$$\Rightarrow z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

② Let S be a regular surface and P be a plane. Suppose $S \cap P = \{P_0\}$ and $P_0 \in \text{int}(S)$. Show that P is tangent to S at P_0.



$\langle x - P_0, \vec{n} \rangle$ does not change sign.



Ans: $\because P_0 \in \text{int}(S)$

$\therefore$ then $\exists$ an open set $U \subseteq \mathbb{R}^2$, $X: U \to S$, a parametrization such that

$$\begin{cases} X(0,0) = P_0, & (0,0) \in U \\ B_\varepsilon(0) \subseteq U & \text{for some } \varepsilon > 0. \end{cases}$$

Let $\vec{n}$ be a unit normal vector of P at P_0.

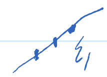
And we consider $f(u,v) = \langle X(u,v) - P_0, \vec{n} \rangle$.

$\therefore f(0,0) = 0$. because $X(0,0) = P_0$.

[claim: $f_u(0,0) = f_v(0,0) = 0$] $\Rightarrow \langle X_u, \vec{n} \rangle_{P_0} = \langle X_v, \vec{n} \rangle_{P_0} = 0 \Rightarrow P$ is the tangent plane of S on the point P_0.

Suppose $f_u(0,0) > 0$, then $\exists\ \varepsilon_1 \in (0, \varepsilon)$ such that

$f(\varepsilon_1, 0) > 0$, $f(-\varepsilon_1, 0) < 0$.



$\because$ f is continuous

$\therefore$ there is a $\varepsilon_2 > 0$ such that

$f(\varepsilon_1, \varepsilon_2) > 0$, $f(-\varepsilon_1, \varepsilon_2) < 0$.

$\therefore$ By IVT, there is a $\varepsilon_0 \in (-\varepsilon_1, \varepsilon_1)$ such that

$$f(\varepsilon_0, \varepsilon_2) = 0$$

$\therefore$ $\langle X(\varepsilon_0, \varepsilon_2) - P_0, \vec{n} \rangle = 0$

$\Rightarrow X(\varepsilon_0, \varepsilon_2) \in P$

this is a contradiction to $S \cap P$ at only one point.

$\therefore$ $f_u(0,0) = f_v(0,0) = 0$. #.

(3) Let S be a regular surface, $P \in \operatorname{int}(S)$, $X: U \to S$ is a parametrization around P.

Show that there is a reparametrization of S around P such that

$$\tilde{g}_{ij}(P) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{h}_{ij}(P) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$ for some real numbers k_1, k_2.

pf: Let S_p be the shape operator at P.

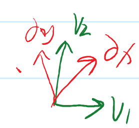
$\because$ $S_p: T_pS \to T_pS$ is linear ($S_p(\vec{v}) = -\frac{\partial N}{\partial \vec{v}}$ for $\vec{v} \in T_pS$).

and T_pS is a two-dimensional linear space ($\cong R^2$).

$\therefore$ there is $\{\vec{v}_1, \vec{v}_2\} \subseteq T_pS$ such that

$$\begin{cases} \|\vec{v}_1\| = \|\vec{v}_2\| = 1 \\ \langle v_1, v_2 \rangle = 0 \\ S_p(v_1) = k_1\vec{v}_1,\ S_p(\vec{v}_2) = k_2\vec{v}_2 \text{ for some } k_1, k_2. \end{cases}$$

(v_1, v_2 are the eigenvectors of S_p).



$\therefore$ $\exists$ a non-singular 2×2 matrix A such that

$$A \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} X_u \\ X_v \end{bmatrix}$$

$\therefore$ Let $(u_0, v_0) \in U$ such that $X(u_0, v_0) = P$.

define $\varphi: R^2 \to U$ by

$$\varphi(\tilde{u}, \tilde{v}) = (A^{-1})^T \left(\begin{bmatrix} \tilde{u} - u_0 \\ \tilde{v} - v_0 \end{bmatrix} \right) + \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

[if $(u_0, v_0) = 0$.

$\varphi = (A^{-1})^T \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}$.

$d\varphi = (A^{-1})^T$.

· Define $Y: R^2 \to S$ by $Y = X \circ \varphi$

$$\begin{cases} \partial_{\tilde{u}} Y = \partial_u X \cdot \frac{\partial \varphi}{\partial \tilde{u}} + \partial_v X \cdot \frac{\partial \varphi}{\partial \tilde{u}} \\ \partial_{\tilde{v}} Y = \partial_u X \cdot \frac{\partial \varphi}{\partial \tilde{v}} + \partial_v X \cdot \frac{\partial \varphi}{\partial \tilde{v}} \end{cases}$$

what we need

$$\begin{cases} \partial_{\tilde{u}} Y = \partial_u X \cdot \frac{\partial \varphi}{\partial \tilde{u}} + \partial_v X \cdot \frac{\partial \varphi}{\partial \tilde{u}} \\ \partial_{\tilde{v}} Y = \partial_u X \cdot \frac{\partial \varphi}{\partial \tilde{v}} + \partial_v X \cdot \frac{\partial \varphi}{\partial \tilde{v}} \end{cases}$$

$$\therefore \begin{bmatrix} \partial_{\tilde{u}} Y \\ \partial_{\tilde{v}} Y \end{bmatrix} = (d\varphi)^T \begin{bmatrix} \partial_u X \\ \partial_v X \end{bmatrix} = A^{-1} \begin{bmatrix} X_u \\ X_v \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}$$

what we need.

$$\therefore \tilde{g}(p) = \begin{bmatrix} \langle \partial_{\tilde{u}} Y, \partial_{\tilde{u}} Y \rangle & \langle \partial_{\tilde{u}} Y, \partial_{\tilde{v}} Y \rangle \\ \langle \partial_{\tilde{u}} Y, \partial_{\tilde{v}} Y \rangle & \langle \partial_{\tilde{v}} Y, \partial_{\tilde{v}} Y \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{h}(p) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

(k_1, k_2 are called the principle curvature of S at p).

Then the Gauss curvature $K = \frac{\det(h)}{\det(g)} = k_1 \cdot k_2$

the mean curvature $H = tr(S) = k_1 + k_2$.