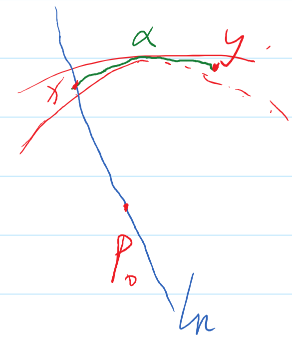
MATH 4030 Tutorial 2

① If all normal lines to a connected surface S passes through a fixed point $P_0 \in \mathbb{R}^3$, show that S is contained in a sphere.



Pf: For any $x, y \in S$, there is a curve $\alpha \subseteq S$ such that α joining x and y

$\alpha'(t) \in T_{\alpha(t)}S$

Consider $f(t) = |\alpha(t) - P_0|^2 = \langle \alpha(t) - P_0, \alpha(t) - P_0 \rangle$

$/\!/ l_n$

$f'(t) = 2\langle \alpha'(t), \alpha(t) - P_0 \rangle = 0$ since $P_0 \in l_n$, $l_n \perp S$

$\therefore$ $f(t)$ is a constant

$\therefore$ S lies on some sphere which is centered at P_0.

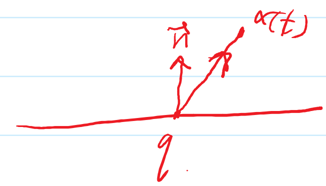
② Let $S \subseteq \mathbb{R}^3$ be a surface.

Suppose $P \subseteq \mathbb{R}^3$ is a plane such that S lies on one side of P, show that $T_q P = T_q S$ for any $q \in P \cap \mathrm{int}(S)$, where $\mathrm{int}(S)$ is the interior of S.

Pf: Let $\vec{n}$ be the unit normal vector of the plane P at q, which is pointing to S.

For any curve $\alpha(t) \subseteq S$ such that $q = \alpha(t_0)$

and we consider $f(t) = \langle \alpha(t) - q, \vec{n} \rangle$



$\therefore$ By the assumption that S lies on one side of P

we have $f(t) \geq 0$

$\because$ $f(t_0) = \langle \alpha(t_0) - q, \vec{n} \rangle = \langle q - q, \vec{n} \rangle = 0$

$\therefore$ $f(t)$ has a local minimum at t_0

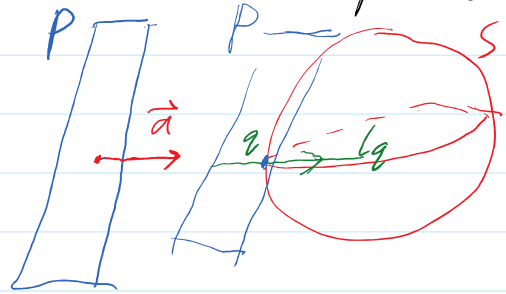
$\therefore$ $f'(t_0) = 0$

$\langle \alpha'(t_0), \vec{n} \rangle = 0$

$\therefore$ $T_q S \perp \vec{n}$

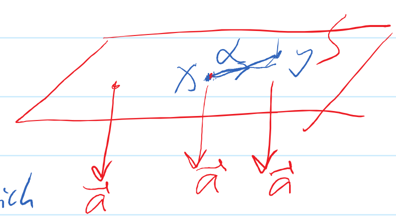
$\therefore$ $T_q S = T_q P$. #.

③ Let S be a closed surface. [if it is compact without boundary]
Let $\vec{a}$ be any unit vector in R^3.
Show that there is a point on S whose normal line is parallel to $\vec{a}$.



Pf: Let P be a plane which is orthogonal to $\vec{a}$ and $P \cap S = \phi$.
$\because$ S is closed.
$\therefore$ We can move the plane P along the direction $\vec{a}$ until the first time P intersec. S at the point q.
$\because$ S lies on one side of the plane P. and $P \perp \vec{a}$.
$\therefore$ $T_q S = T_q P$
$\because$ $\vec{a} \perp P$
$\therefore$ $\vec{a} \perp T_q S$ #.

④ Let S be a connected surface and $\vec{a}$ be a unit vector in R^3.
Suppose all the normal lines of S are parallel to $\vec{a}$.
Show that S is contained in a plane.



Pf: Choose a point $x \in S$, we need to prove that $\langle y-x, \vec{a} \rangle = 0$ for any $y \in S$.
$\therefore$ for any $y \in S$, Let α be a regular curve which lies on S and $\alpha(0)=x$, $\alpha(1)=y$.
Let $f(t) = \langle \alpha(t) - x, \vec{a} \rangle$

$$\begin{cases} f(0) = \langle x-x, \vec{a} \rangle = 0 \\ f'(t) = \langle \alpha'(t), \vec{a} \rangle = 0 \end{cases}$$ by the assumption. $[\alpha'(t) \perp \vec{a}]$.

$\therefore$ $f(t) \equiv 0$
$\therefore$ $\langle y-x, \vec{a} \rangle = 0$
$\because$ y is an arbitrary point on S
$\therefore$ S lies on a plane which is orthogonal to $\vec{a}$. #