MATH 4030 Tutorial 1

① Reparametrize the following curves by arc-length.

(a) $\alpha: R \to R^2: t \mapsto (r\cos kt,\ r\sin kt)$, $k, r > 0$

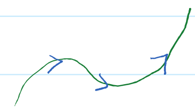
Ans: $\alpha'(t) = (-kr\sin kt,\ kr\cos kt)$

$|\alpha'(t)| = kr$

$s(t) = \int_0^t |\alpha'(\tau)|d\tau = krt$

$\therefore\ t = \frac{1}{kr}s$

$\beta(s) = \alpha(t(s)) = (r\cos\frac{s}{r},\ r\sin\frac{s}{r})$



$\beta(s) = (r\cos\frac{s}{r},\ r\sin\frac{s}{r})$

$\beta'(s) = (-\sin\frac{s}{r},\ \cos\frac{s}{r})$

(b) $\alpha: R \to R^3: t \mapsto (a\cos t,\ a\sin t,\ bt)$

Ans: $\alpha'(t) = (-a\sin t,\ a\cos t,\ b)$

$|\alpha'(t)| = \sqrt{a^2+b^2}$

$s(t) = \int_0^t |\alpha'| = \sqrt{a^2+b^2}\cdot t$

$t = \frac{s}{\sqrt{a^2+b^2}}$

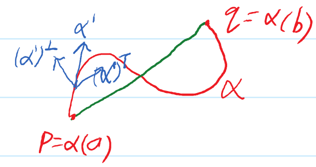
$\beta(s) = \alpha(t(s)) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ b\cdot\frac{s}{\sqrt{a^2+b^2}}\right)$

② Let $\alpha: I \to R^3$ be a curve and $[a,b] \subseteq I$, prove that

$$|\alpha(a) - \alpha(b)| \leq L_a^b(\alpha)$$

In other words, straight lines are the shortest curves joining two given points.

Ans: let $\vec{n} = \frac{\alpha(b) - \alpha(a)}{|\alpha(b) - \alpha(a)|}$ [if $\alpha(b) = \alpha(a)$, the equality holds]

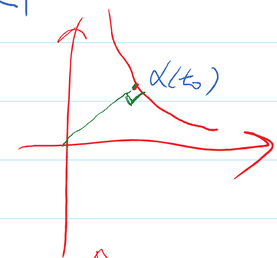


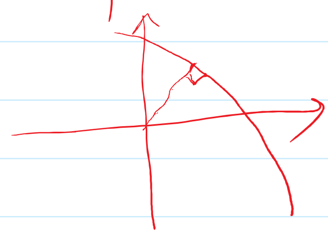
$|\alpha(b) - \alpha(a)| = \int_a^b \langle \alpha'(t), \vec{n}\rangle dt$ $\left(= \langle \int_a^b \alpha'(t), \vec{n}\rangle = \langle \alpha(b) - \alpha(a), \vec{n}\rangle\right)$

$$\le \int_a^b |\alpha'(t)| \cdot |\vec{n}|\, dt$$
$$= \int_a^b |\alpha'(t)| = L_a^b(\alpha) \quad \#.$$

③ Let $\alpha: I \to R^3$ be a curve which does not pass through the origin (i.e. $\alpha(t) \neq 0$ for all $t \in I$).

If $\alpha(t_0)$ is the point which is $\boxed{\text{closest to the origin}}$ and $\alpha'(t) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.





Ans: define $f(t) = |\alpha(t)|^2$
$= \langle \alpha(t), \alpha(t) \rangle$

$\because$ $\alpha(t_0)$ is the closest point to the origin

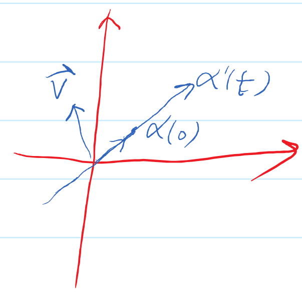
$\therefore$ $f(t_0)$ is a local minimal.

$\therefore$ $0 = f'(t_0) = 2\langle \alpha'(t_0), \alpha(t_0) \rangle$

$\because$ $\alpha'(t_0) \neq 0$, $\alpha(t_0) \neq 0$

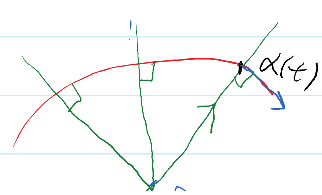
$\therefore$ $\alpha'(t_0) \perp \alpha(t_0)$

④ Let $\alpha: I \to R^3$ be a curve and $\vec{V} \in R^3$ be a fixed vector.
Assume that $\langle \alpha'(t), V \rangle = 0$ for all $t \in I$.
and $\langle \alpha(0), V \rangle = 0$
Prove that $\langle \alpha(t), V \rangle = 0$ for all $t \in I$.



Ans: define $f(t) = \langle \alpha(t), \vec{V} \rangle$
then $f(0) = 0$
$f'(t) = \langle \alpha'(t), \vec{V} \rangle = 0$
$\therefore$ $f(t) \equiv 0$. $\#$.

⑤ Let $\alpha: I \to R^2$ be a regular curve such that all the normal lines pass through a fixed point $P \in R^2$.
Prove that the trace $\alpha(I)$ is contained in a circle of some radius $r > 0$ centered at P.



Pf: α is regular
$\therefore$ $\alpha'(t) \neq 0$
$\therefore$ $\alpha(t) - \vec{P} \perp \alpha'(t)$

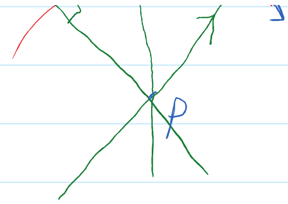
$\because\ \alpha'(t) \neq 0$

$\because\ \alpha(t) - \vec{P} \perp \alpha'(t)$

$\therefore\ \langle \alpha'(t), \alpha(t) - \vec{P} \rangle = 0$ for all $t \in I$

$\therefore\ \langle \alpha(t) - \vec{P}, \alpha(t) - \vec{P} \rangle' = 0$ for all $t \in I$

$\therefore\ |\alpha(t) - \vec{P}| = r$ for some non-negative r.



if $r = 0$, then $\alpha(t)$ is just one point. $\Rightarrow\Leftarrow$

$\therefore\ r > 0$, then $\alpha(t)$ lies on the circle with radius r, centered at P.

⑥
$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$