Hopf's Umlaufsatz

Theorem (Hopf's Umlaufsatz, Theorem of Turning Tangents)

Let $\alpha : [0, l] \to \mathbb{R}^2$ be a piecewise regular, simple closed curve with $\alpha(0) = \alpha(l)$. Let $\alpha(t_1), \dots, \alpha(t_k)$, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$ be the vertices of α with exterior angle θ_i. Let φ_i be smooth choice of angles defined in $[t_i, t_{i+1}]$ such that the oriented angle from the positive axis to $\alpha'(t)$ is $\varphi_i(t)$ (i.e. $\alpha' = (\cos \varphi_i(t), \sin \varphi_i(t))$) for $t \in [t_i, t_{i+1}]$. Then

$$\sum_{i=1}^{k} (\varphi_i(t_{i+1}) - \varphi_i(t_i)) + \sum_{i=0}^{k} \theta_i = \pm 2\pi.$$

It is $+1$ if α is positively oriented and -1 if it is negatively oriented, with respect to the usual orientation of $\mathbb{R}^2$.

Proof for smooth curves

Proof Let us prove the theorem for smooth simple closed curve. We assume that α is positively oriented.

Step 1: We may assume that $\alpha(0) = \alpha(l)$ is the origin. Moreover, we may assume that there is a unit vector $\mathbf{u}$ so that $p + t\mathbf{u}$ will not intersect α, for $t > 0$. We may also assume that $\alpha'(l) = \alpha'(0) = \mathbf{e}_1 = (1, 0)$. Then $\mathbf{e}_2 = (0, 1)$ is pointing inside α because it is positively oriented. Then $\mathbf{u}$ is pointing downward.

Step 2: Let T be the triangle:

$$T = \{(s, t) |\ 0 \le s \le t \le l\}.$$

Define a map $\mathbf{v} : T \to \mathbb{S}^2$ as follows

$$\mathbf{v}(s,t) = \begin{cases} \frac{\alpha(t)-\alpha(s)}{|\alpha(t)-\alpha(s)|}, & \text{if } s < t \text{ and } (s,t) \ne (0,l)\ ; \\ \alpha'(s), & \text{if } s = t\ ; \\ -\alpha'(0), & \text{if } (s,t) = (0,l). \end{cases}$$

Then **v** well-defined because the curve is simple closed. It is continuous. in fact:
(i) If $s_0 < t_0$ and $(s_0, t_0) \neq (0, 1)$, then it is obvious.
(ii) If $s_0 = t_0$, then it is also true.
(iii) At $(0, l)$, then for (s, t) near this point, $s < t$ and

$$\mathbf{v}(s,t) = \frac{\alpha(t) - \alpha(s)}{|\alpha(t) - \alpha(s)|} \to -\alpha'(l) = -\alpha'(0) = -\mathbf{e}_1.$$

Step 3: There is a continuous function θ on T so that

$$\mathbf{v}(s,t) = (\cos(s,t), \sin(s,t)).$$

Then the theorem is proved if one can show that

$$\theta(l,l) - \theta(0,0) = 2\pi$$

if α is positively oriented. We may also choose that $\theta(0,l) = \pi$.

Step 4: Note that

$$\theta(l,l)-\theta(0,0)=\theta(l,l)-\theta(0,l)+\theta(0,l)-\theta(0,0).$$

Now

$$\theta(l,l)-\theta(0,l)=\lim_{s\to l}(\theta(s,l)-\theta(0,l)).$$

For $0 < s < l$, $\theta(s,l) - \theta(0,l)$ measures the angle between $\mathbf{v}(s,l)$ and $-\mathbf{e}_1$. Note that $\mathbf{v}(s,l) \neq -\mathbf{u}$. So $|\theta(s,l) - \theta(0,l)| < 2\pi$. But

$$\mathbf{v}(l,l) = \mathbf{e}_1.$$

Hence

$$\theta(l,l) - \theta(0,l) = \pm\pi.$$

If α is positively oriented, then the curve must be above the x-axis as the unit normal is pointing insider the curve. We should get π. Similarly, $\theta(0,l) - \theta(0,0) = \pi$.

Jordan curve theorem

Theorem (Jordan curve theorem)

Let α be a continuous simple closed curve in $\mathbb{R}^2$ (or in $\mathbb{S}^2$), then α will separate $\mathbb{R}^2$ (or $\mathbb{S}^2$) into two components (i.e. open connected sets).

Proof for Jordan polygon

Let P be a Jordan polygon.

Step 1: Choose a direction which is not parallel to any side, given by a unit vector $\mathbf{u}$. This can be done because P has only finitely many sides. Let $p \notin P$, the straight line $l : \mathbf{p} + t\mathbf{u}$, will intersect a side of P at most once.

Step 2: For any point $\mathbf{p} \notin P$, let $n(\mathbf{p})$ be the number of points of intersection of the ray $\mathbf{p} + t\mathbf{u}$, $t > 0$ with P, with the following convention. If $\mathbf{p} + t\mathbf{u}$ passes through a vertex, then it will be counted as a point of intersection only if the two sides with this vertex is on the different side of the ray. By **Step 1**, $n(\mathbf{p})$ is finite.

Step 3: Let

$$\mathcal{E} = \{\mathbf{p} \notin P | \ n(\mathbf{p}) \text{ is even}\},$$

$$\mathcal{O} = \{\mathbf{p} \notin P | \ n(\mathbf{p}) \text{ is odd}\}.$$

Step 4: $\mathcal{E} \neq \emptyset$, and $\mathcal{O} \neq \emptyset$. In fact, let l be any ray $\mathbf{p} + t\mathbf{u}$. Then this ray will not intersect P for $t \geq t_0$, say. Let $q = \mathbf{p} + t_0\mathbf{u}$. Then $q + t\mathbf{u}$ will not intersect P. So $q \in \mathcal{E}$. There is $p \notin P$ so that $\mathbf{p} + t\mathbf{u}$, $t > 0$ will meet a side P . Let $q = \mathbf{p} + t_0\mathbf{u}$ be the last point that of intersection. Then the point just before q will be in $\mathcal{O}$.

Step 5: If l is a line segment in $\mathbb{R}^2 \setminus P$, then the parity is constant. For the parity of a point moving along such a segment can only change when the ray in the fixed direction $\mathbf{u}$ through the point passes through a vertex of P, and in neither of the two possible cases will the parity actually change, because of the agreement made in the preceding paragraph. Hence both $\mathcal{E}$ and $\mathcal{O}$ are open. Moreover, any point in $\mathcal{E}$ cannot be joined to $\mathcal{O}$ by a polygonal path without intersecting P.

Step 6: If two points $\mathbf{p}, \mathbf{q}$ not in P are close to a segment of P and is on different sides of the segments, then they belongs to different classes of parity.

Step 7: Using the fact that P is a Jordan curve and **Step 6** to prove that if $\mathbf{p}, \mathbf{q}$ have the same parity with $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2 \setminus P$, then they can be connected by a polygonal path inside $\mathbb{R}^2 \setminus P$. Hence $\mathbb{R}^2 \setminus$ has at most two components.

In higher dimension, we have:

Theorem

(Jordan-Brouwer Separation Theorem) *Let* $B \subset \mathbb{S}^n$ *which is homeomorphic to* $\mathbb{S}^{n-1}$, $n \geq 2$. *Then* $\mathbb{S}^n \setminus B$ *consists of two arcwise connected components such that* B *is the boundary of each component.*