Hyperbolic plane $\mathbb{H}^2$

The hyperbolic plane is given as follows. It is an abstract surface. Let

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | \ y > 0\}$$

so that the first fundamental form is

$$g_{ij}(x, y) = \frac{1}{y^2}.$$

That is to say, if $\alpha(t) = (x(t), y(t))$ is a regular curve, then

$$\langle \alpha', \alpha' \rangle = g_{11}(x')^2 + 2g_{12}x'y' + g_{22}(y')^2 = \frac{1}{y^2(t)} \left((x')^2 + (y')^2 \right).$$

Gaussian curvature of $\mathbb{H}^2$

To compute the Gaussian curvature.

By previous result

$$K = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} \right) \log y = -1.$$

Completeness of $\mathbb{H}^2$

$\mathbb{H}^2$ is complete: every divergent path has infinite length.
Let $\alpha : [0, 1)$ be a divergent path. Let $\alpha(t) = (x(t), y(t))$, then the length of α is given by

$$\ell(\alpha) = \int_0^1 \frac{1}{y(t)} \left((x')^2(t) + (y')^2(t)\right)^{\frac{1}{2}} dt.$$

Since α is divergent, there exist $t_k \to 1$ so that (i) $y(t_k) \to 0$; (ii) $y(t_k) \to \infty$; or (iii) $b > y(t_k) > a > 0$ for some a, b and $x(t_k) \to \infty$. In any case we will have $\ell(\alpha) = \infty$.

Geodesics of $\mathbb{H}^2$

Consider the energy functional $\int \mathcal{L} dt$, with

$$\mathcal{L} = \frac{1}{2} y^{-2} \left((\dot{x})^2 + (\dot{y})^2 \right).$$

To find the E-L equations, first we have

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 0; \\ \frac{\partial \mathcal{L}}{\partial \dot{x}} = y^{-2} \dot{x}; \\ \frac{\partial \mathcal{L}}{\partial y} = -y^{-3} \left((\dot{x})^2 + (\dot{y})^2 \right); \\ \frac{\partial \mathcal{L}}{\partial \dot{y}} = y^{-2} \dot{y}. \end{cases}$$

Hence the E-L equations are:

$$\begin{cases} 0 = \dfrac{\partial \mathcal{L}}{\partial x} - \dfrac{d}{dt}(\dfrac{\partial \mathcal{L}}{\partial \dot{x}}) = -\dfrac{d}{dt}(y^{-2}\dot{x}); \\ 0 = \dfrac{\partial \mathcal{L}}{\partial y} - \dfrac{d}{dt}(\dfrac{\partial \mathcal{L}}{\partial \dot{y}}) = -y^{-3}\left((\dot{x})^2 + (\dot{y})^2\right) - \dfrac{d}{dt}(y^{-2}\dot{y}). \end{cases}$$

Geodesics of $\mathbb{H}^2$, cont.

That is:

$$\begin{cases} \ddot{x} - 2y^{-1}\dot{x}\dot{y} = 0; \\ \ddot{y} + y^{-1}\left((\dot{x})^2 - (\dot{y})^2\right) = 0. \end{cases}$$

Obviously, $x =$constant is a geodesic. Next consider the semi-circle: $x = R\cos t, y = R\sin t$, $0 < t < \pi$. Then

$$x'' - 2y^{-1}x'y' = -\frac{\cos t}{\sin t}x'$$

and

$$y'' - y^{-1}\left((x')^2 + (y')^2\right) = -\frac{\cos t}{\sin t}y'.$$

So it is a pre-geodesic.

Proposition

The geodesics are either $\{x = constant\}$ or semicircles with centers at the x-axis and orthogonal to the x-axis.

Isometry of $\mathbb{H}^2$

Let $F : \mathbb{H}^2 \to \mathbb{H}^2$ defined as

$$F(x, y) = (u, v),$$

so that

$$u + \mathbf{i}v = \frac{az + b}{cz + d}$$

with a, b, c, d real and $ac - bd = 1$. Here $z = x + \mathbf{i}y$.
Then

$$u + \mathbf{i}v = \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2}$$

Imaginary part of $(az + b)(c\bar{z} + d)$ is

$$(ad - bc)y = y > 0.$$

So this is well-defined.

Isometry of $\mathbb{H}^2$, cont.

Let $\alpha(t) = (x(t), y(t))$ be a curve in $\mathbb{H}^2$, $t \in [0,1]$. Then the length of $\alpha(t)$ is given by

$$\ell(\alpha) = \int_0^1 \frac{1}{y(t)} \left((x')^2(t) + (y')^2(t)\right)^{\frac{1}{2}} dt = \int_0^1 \frac{1}{y(t)} |z'(t)| dt.$$

Let $\beta(t) = F \circ \alpha$. Write $\beta(t) = (u(t), v(t))$, then the length of β is

$$\ell(\beta) = \int_0^l \frac{1}{v(t)} |w'(t)| dt$$

where $w = u + \mathbf{i}v$.

Isometry of $\mathbb{H}^2$, cont.

Now

$$v = \frac{y}{|cz+d|^2}.$$

$$\begin{aligned} w' &= \left(\frac{az+b}{cz+d}\right)' \\ &= \frac{az'(cz+d) - cz'(az+b)}{(cz+d)^2} \\ &= \frac{z'}{(cz+d)^2}. \end{aligned}$$

So

$$\frac{1}{v(t)}|w'(t)| = \frac{1}{y(t)}|z'(t)|.$$

What is the image of $x =$constant. That is the image of $F(0, y)$.

$$u + \mathbf{i}v = F(0, y) = \frac{a\mathbf{i}y + b}{c\mathbf{i}y + d}.$$

It is a circle with centered at $\frac{1}{2}(a/c + b/d)$ if $c \neq 0, d \neq 0$.

Geodesic equations of surfaces of revolution

Consider the surface of revolution given by

$$\mathbf{X}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

with $f > 0$. In the following f' means $\frac{df}{dv}$, etc. If there is come confusion, we will write f_v instead, etc.
Consider $u^1 \leftrightarrow u, u^2 \leftrightarrow v$.

$$\begin{cases} g_{11} = & E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle = f^2, ; \\ g_{12} = & g_{21} = F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0 \\ g_{22} = & G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle = (f')^2 + (g')^2. \end{cases}$$

So

$$\begin{cases} \Gamma^1_{11} = 0, \Gamma^1_{12} = \frac{f'}{f}, \Gamma^1_{22} = 0; \\ \Gamma^2_{11} = -\frac{ff'}{(f')^2+(g')^2}, \Gamma^2_{12} = 0, \Gamma^2_{22} = \frac{f'f''+g'g''}{(f')^2+(g')^2}. \end{cases}$$

Geodesic equations of surfaces of revolution

Hence geodesic equations are:

$$\begin{cases} \ddot{u} + \frac{2f'}{f}\dot{u}\dot{v} = 0; \\ \ddot{v} - \frac{ff'}{(f')^2+(g')^2}(\dot{u})^2 + \frac{f'f''+g'g''}{(f')^2+(g')^2}(\dot{v})^2 = 0. \end{cases}$$

Corollary

Any meridian is a geodesic. A parallel $\mathbf{X}(u, v_0)$ *is a geodesic if and only if* $f'(v_0) = 0$.

To study the behavior of general geodesics, we begin with the following lemma:

Lemma

Let $a_1(t), a_2(t)$ be smooth functions on $(T_1, T_2) \subset \mathbb{R}$ such that $a_1^2 + a_2^2 = 1$. For any $t_0 \in (T_1, T_2)$ and θ_0 such that $a_1(t_0) = \cos\theta_0$, $a_2(t_0) = \sin\theta_0$, there exists unique a smooth function $\theta(t)$ with $\theta(t_0) = \theta_0$ such that $a_1(t) = \cos\theta(t)$ and $a_2(t) = \sin\theta(t)$.

Proof of the lemma

Proof: Suppose θ satisfies the condition. Then $a_1' = -\theta' \sin\theta$, $a_2' = \theta' \cos\theta$. Hence $\theta' = a_1 a_2' - a_2 a_1'$. From this we have uniqueness. To prove existnce, fix $t_0 \in (T_1, T_2)$ and let θ_0 be such that $\cos\theta_0 = a_1(0)$, $\sin\theta_0 = a_2(0)$. Let

$$\theta(t) = \theta_0 + \int_{t_0}^{t} (a_2' a_1 - a_1' a_2) d\tau.$$

Let $f = (a_1 - b_1)^2 + (a_2 - b_2)^2$, where $b_1 = \cos\theta, b_2 = \sin\theta$. Then $f = 2 - 2a_1 b_1 - 2a_2 b_2$.

Proof of lemma, cont.

Then

$$\begin{aligned}
-\frac{1}{2}f' =& a_1' b_1 + a_1 b_1' + a_2' b_2 + a_2 b_2' \\
=& a_1' b_1 - \theta' a_1 b_2 + a_2' b_2 + \theta' a_2 b_1 \\
=& (a_2' a_1 - a_1' a_2)(-a_1 b_2 + a_2 b_1) + a_1' b_1 + a_2' b_2 \\
=& - a_1^2 a_2' b_2 + a_2 a_2' a_1 b_1 + a_1 a_1' a_2 b_2 - a_2^2 a_1' b_1 + a_1' b_1 + a_2' b_2 \\
=& - a_1^2 a_2' b_2 - a_1 a_1' a_1 b_1 - a_2 a_2' a_2 b_2 - a_2^2 a_1' b_1 + a_1' b_1 + a_2' b_2 \\
=& 0
\end{aligned}$$

because $a_1^2 + a_2^2 = 1$ and $a_1 a_1' + a_2 a_2' = 0$.

General geodesics, cont.

Now let $\alpha(s) = \mathbf{X}(u(s), v(s))$ be a geodesic on M parametrized by arc length. Let $\mathbf{e_1} = \mathbf{X}_u/|\mathbf{X}_u|$ and $\mathbf{e_2} = \mathbf{X}_v/|\mathbf{X}_v|$. Then $\mathbf{e}_1, \mathbf{e}_2$ are orthonormal. Let

$$\alpha' = a_1\mathbf{e_1} + a_2\mathbf{e_2}.$$

By the lemma there exists smooth function $\theta(s)$ such that $a_1 = \sin\theta$, $a_2 = \cos\theta$. Note that θ *is the angle between* α' *and the meridian*. That is:

$$\sin\theta = \langle \alpha', \mathbf{e_1} \rangle = f\dot{u}.$$

Clairaut's Theorem

Proposition (Clairaut's Theorem)

$r(s)\sin\theta(s)$ is constant along α, where $r(s)$ is the distance of $\alpha(s)$ from the z-axis.

Proof.

Denote the $\frac{d\alpha}{ds}$ by α' etc. Since $r(s) = f(v(s))$,

$$r' = f_v v'.$$

Also $\sin\theta = \langle \alpha', \mathbf{e_1} \rangle = u'f$, so $(\sin\theta)' = u''f + u'v'f_v$.

$$\begin{aligned}(r\sin\theta)' =& f_v v' u' f + u'' f + f_v u' v' \\ =& f\left(u'' + \frac{2f_v}{f} u' v'\right) \\ =& 0.\end{aligned}$$

□

Nöether's first theorem, a digression

Consider the action:

$$S = \int_a^b \mathcal{L}(\phi, \dot{\phi}, t) dt.$$

here $\phi = (\phi^k)$. To be precise, we denote $\mathcal{L}(u, v, t)$ so that in the above, $u^k = \phi^k, v^k = \dot{\phi}^k$. Consider the transformation:

$$\begin{cases} \widetilde{t} = \widetilde{t}(\phi, t, \epsilon) \\ \widetilde{\phi} = \widetilde{\phi}(\phi, t, \epsilon). \end{cases}$$

Let $\widetilde{\mathcal{L}} = \mathcal{L}(\widetilde{\phi}, \widetilde{\phi}', \widetilde{t})$. $'$ means derivative w.r.t. $\widetilde{t}$. Assume $\widetilde{t} = t, \widetilde{\phi}(\phi, t, \epsilon) = \phi$ at $\epsilon = 0$.

Nöether's first theorem, cont.

The action is said to be invariant if

$$\mathcal{L} - \widetilde{\mathcal{L}}\frac{d\widetilde{t}}{dt} = O(\epsilon^2).$$

Hence we have

$$\frac{d}{d\epsilon}\left(\mathcal{L} - \widetilde{\mathcal{L}}\frac{d\widetilde{t}}{dt}\right)\Big|_{\epsilon=0} = 0.$$

Let

$$\begin{cases} \tau = \frac{\partial \widetilde{t}}{\partial \epsilon}\Big|_{\epsilon=0}; \\ \xi^k = \frac{\partial \widetilde{\phi}}{\partial \epsilon}\Big|_{\epsilon=0}. \end{cases}$$

Proof is given at the end of the note for your reference.

Theorem (Nöether)

If $\mathcal{L}$ is invariant in the above sense, then

$$\frac{d}{dt}\left(\tau H - p_k \xi^k\right) = (\xi^k - \tau \dot{\phi}^k) E_k$$

where $p_k = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}$, $H = p_k \dot{\phi}^k - \mathcal{L}$ *and* E_k *is the E-L expression*

$$\frac{\partial \mathcal{L}}{\partial \phi^k} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}\right).$$

In particular, if $E_k = 0$, *then* $\tau H - p_k \xi^k$ *is conserved, i.e. constant along the solution.*

Consider m particles in three space with coordinates (x^j, y^j, z^j) with mass M_j. Then

$$\mathcal{L} = \frac{1}{2}\sum_j M_j \left[(\dot{x}^j)^2 + (\dot{y}^j)^2 + (\dot{z}^j)^2\right] - V(t, x, y, z).$$

Then

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}$$

which is $M_j \dot{x}^j$, etc.

$$\begin{aligned} H =& M_j \dot{x}^j \dot{x}^j - \frac{1}{2}\sum_j M_j \left[(\dot{x}^j)^2 + (\dot{y}^j)^2 + (\dot{z}^j)^2\right] - V(t, x, y, z) \\ =& \frac{1}{2}\sum_j M_j \left[(\dot{x}^j)^2 + (\dot{y}^j)^2 + (\dot{z}^j)^2\right] + V(t, x, y, z). \end{aligned}$$

Suppose V does not depend on t (i.e. there is only one independent variable). Then the transformation $t \to t + \epsilon$, with x, y, x fixed is invariant. We have $\tau = 1$, $\xi = 0$.

Hence we have,

$$0 = \frac{d}{dt}\left(-\mathcal{L} + \sum_j M_j\left((\dot{x}^j)^2 + (\dot{y}^j)^2 + (\dot{z}^j)^2\right)\right) = \frac{d}{dt}(K + V).$$

Here K is the kinetic energy. So the total energy is preserved.

Geodesics of surfaces of revolution, cont.

Clairaut's Theorem revisited: In this case for the energy functional,

$$\mathcal{L} = \frac{1}{2}(f^2(\dot{u})^2 + (f_v^2 + g_v^2)(\dot{v})^2).$$

If we let $\widetilde{u} = u + \epsilon$, $\widetilde{v} = v$, $\widetilde{t} = t$, then

$$\mathcal{L} = \widetilde{\mathcal{L}}$$

Here we have $\tau = 0$, $\xi^1 = 1, \xi^2 = 0$.

$$p_1 = f^2 \dot{u}$$

So, we have

$$\frac{d}{dt}(f^2 \dot{u}) = 0$$

along the geodesic.

In fact, this is just one of the E-L equations. Note that

$$\sin\theta = \langle \alpha', \mathbf{e}_1 \rangle = \langle \mathbf{X}_u \dot{u} + \mathbf{X}_v \dot{v}, \frac{\mathbf{X}_u}{|\mathbf{X}_u|} \rangle = f \dot{u}.$$

So $r(s)\sin\theta(s) = f(\alpha(s))\sin\theta(s) = f^2 \dot{u}$.

Let us analyse a geodesic $\alpha(s)$, $0 \leq s < L \leq \infty$, on the surface of revolution parametrized by arc length. Let us assume that

$g(v)$ is increasing, i.e. $g_v > 0$.

Let $r(s)$ and $\theta(s)$ be as in Clairaut's Theorem. Let $\theta_0 = \theta(0)$. We may assume that

$0 \leq \theta_0 \leq \frac{\pi}{2}$.

By Clairaut's Theorem,

$r(s) \sin \theta(s) = R$ for some constant $R \geq 0$.

Note that $r(s) \geq R$.
Case 1: $R = 0$, then $\theta = \pi/2$ along α. So it is a meridian.
Case 2: $R > 0$. Then $\sin\theta_0 = R/r(0) < 1$. So $0 \leq \theta_0 < \pi/2$. So α is going up. Then we must have $f_v(\alpha(0)) \neq 0$. Let us assume that $f_v(\alpha(0)) > 0$. Hence initially, $r(s) = f(\alpha(s)) > R$ and it continue to go up. We consider two cases:

Case 2(i) There is no parallel above $\alpha(0)$ with radius R. Then α will go up all the way.
Case 2(ii) There is a parallel above $\alpha(0)$ so that the radius is R. Let C be the first one above $\alpha(0)$. Then we have two more subcases:
(ii)(a) C is a geodesic. Then α will approach to C but never intersect C.
(ii)(b) C is not a geodesic, then α will touch C and bounces away.

To summarize, in the above settings, we have:

Proposition

(i) *If $R = 0$, then α is a meridian.*

(ii) *$R > 0$. Then geodesic will go up for all s, as long as $r > R$, i.e. the z coordinate of α is increasing in s. Either α does not come close to any parallel of radius R, and α will go up for all s, or α will be close to a parallel C of radius R. Let C be the first such parallel above α. Then we have the following cases:*

   (a) *C is a geodesic. Then α will not meet C and α will come arbitrarily close to C without intersecting C.*

   (b) *C is not a geodesic. Then there is $\alpha(s_0) \in C$ for some s_0 and α will bounce off from C and will turn downward.*

Proof of Nöether's theorem

Proof: At $\epsilon = 0$,

$$0 = \frac{d}{d\epsilon}\left(\mathcal{L} - \tilde{\mathcal{L}}\frac{d\tilde{t}}{dt}\right)$$
$$= -\frac{\partial}{\partial\epsilon}\tilde{\mathcal{L}} - \mathcal{L}\frac{\partial}{\partial\epsilon}(\frac{d\tilde{t}}{dt})$$
$$= -\frac{\partial\mathcal{L}}{\partial\phi^k}\xi^k - \frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\frac{\partial\dot{\tilde{\phi}}^k}{\partial\epsilon} - \frac{\partial\mathcal{L}}{\partial t}\tau - \mathcal{L}\dot{\tau}$$

Now

$$\frac{\partial\dot{\tilde{\phi}}^k}{\partial\epsilon} = \frac{\partial}{\partial\epsilon}(\frac{\partial\dot{\tilde{\phi}}^k}{dt}\cdot\frac{dt}{d\tilde{t}})$$
$$= \dot{\xi}^k - \dot{\phi}^k\dot{\tau}.$$

So we have

$$0 = \frac{\partial \mathcal{L}}{\partial \phi^k}\xi^k + \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}\left(\dot{\xi}^k - \dot{\phi}^k\dot{\tau}\right) + \frac{\partial \mathcal{L}}{\partial t}\tau + \mathcal{L}\dot{\tau}$$

Also,

$$
\begin{aligned}
\tau \frac{\partial}{\partial t}\mathcal{L} =& \tau \frac{d}{dt}\mathcal{L} - \tau \frac{\partial \mathcal{L}}{\partial \phi^k}\dot{\phi}^k - \tau \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}\ddot{\phi}^k \\
=& \tau \frac{d}{dt}\mathcal{L} - \tau \frac{\partial \mathcal{L}}{\partial \phi^k}\dot{\phi}^k - \frac{d}{dt}\left(\tau \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}\dot{\phi}^k\right) + \frac{d}{dt}\left(\tau \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}\right)\dot{\phi}^k
\end{aligned}
$$

So

$$
\begin{aligned}
-\frac{d}{dt}&\left(\tau\mathcal{L}-\tau\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\dot{\phi}^k\right)\\
=&\frac{\partial\mathcal{L}}{\partial\phi^k}\xi^k+\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\left(\dot{\xi}^k-\dot{\phi}^k\dot{\tau}\right)-\tau\frac{\partial\mathcal{L}}{\partial\phi^k}\dot{\phi}^k+\frac{d}{dt}\left(\tau\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\right)\dot{\phi}^k\\
=&(\xi^k-\tau\dot{\phi}^k)\frac{\partial\mathcal{L}}{\partial\phi^k}+\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\left(\dot{\xi}^k-\dot{\phi}^k\dot{\tau}\right)+\frac{d}{dt}\left(\tau\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\right)\dot{\phi}^k\\
=&(\xi^k-\tau\dot{\phi}^k)\frac{\partial\mathcal{L}}{\partial\phi^k}+\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\dot{\xi}^k+\tau\dot{\phi}^k\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\\
=&(\xi^k-\tau\dot{\phi}^k)\frac{\partial\mathcal{L}}{\partial\phi^k}+\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\xi^k\right)-\xi^k\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\right)+\tau\dot{\phi}^k\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\\
=&(\xi^k-\tau\dot{\phi}^k)\left(\frac{\partial\mathcal{L}}{\partial\phi^k}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\right)+\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}^k}\xi^k\right).
\end{aligned}
$$

So we have

$$\frac{d}{dt}\left(\tau H - p_k \xi^k\right) = (\xi^k - \tau\dot{\phi}^k)\left(\frac{\partial \mathcal{L}}{\partial \phi^k} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}\right)$$

where

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}, \quad H = p_k \dot{\phi}^k - L$$