Definition

A triangulation of a compact surface M consists of a finite family of closed subsets $\{T_1, T_2, \ldots, T_n\}$ that cover M, and a family of homeomorphisms

$$\phi_i : T_i' \to T_i$$

$i = 1, \ldots, n$ where each T_i' is a triangle in the plane $\mathbb{R}^2$. (i.e., a compact subset of $\mathbb{R}^2$ bounded by three distinct straight lines). The subsets T_i are called "triangles." The subsets of T_i that are the images of the vertices and edges of the triangle T_i' under ϕ_i are also called vertices and edges, respectively. It is required that any two distinct triangles, T_i and T_j are either disjoint, have a single vertex in common, or have one entire edge in common.

Facts

- Any compact surface has a triangulation.
- If M is a compact surface with boundary, then one can find a triangulation so that:

  (i) No edge has both vertices contained in the boundary unless the entire edge is contained in the boundary;
  (ii) no triangle has more than one edge contained in the boundary;
  (iii) if T_i and T_j are triangles each of which has one edge contained in the boundary, then they are either disjoint or have one vertex in common at the boundary.

Euler Characteristic

Let be a compact surface with triangulation $\{T_1, \ldots, T_n\}$. Let

$$\begin{cases} V = \text{total number of vertices;} \\ E = \text{ total number of edges;} \\ F = \text{total number of triangles (faces).} \end{cases}$$

In this case $F = n$. Then the Euler characteristic of M is defined as

$$\chi(M) = V - E + F.$$

Fact: Let $\{\mathcal{T}\}$ be a triangulation of M and let $\{\mathcal{T}'\}$ be its barycentric subdivision. Let V, E, F be the number of vertices, edges, faces of $\{\mathcal{T}\}$ and let V', E', F' be the number of vertices, edges, faces of $\{\mathcal{T}'\}$. Then

$$V - E + F = V' - E' + F'.$$

Theorem

$\chi(M)$ *does not depend on the triangulation. So it is well-defined.*

Classification of compact oriented surfaces without boundary

Proposition

Every compact region in a regular surface with piecewise smooth boundary has a triangulation so that each triangle is inside an isothermal coordinate neighborhood. If each triangle is positively oriented, then adjacent triangles determine opposite orientation at the common edge.

Theorem

Every oriented compact surface M without boundary is homeomoprhic to the unit sphere, or the unit sphere with g handles attached. Moreover,

$$\chi(M) = 2 - 2g$$

where g is the number of handles.

See W.S. Massey: Algebraic topology: an introduction.

The Gauss-Bonnet Theorem: global version

Theorem

Let M be an oriented regular surface and $\mathcal{R}$ is a region in M bounded by piecewise smooth simply closed curve $C_1, \ldots, C_n$ which are positively oriented. Let $\theta_1, \ldots, \theta_l$ be the set of exterior angles of $\mathcal{R}$. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g ds + \iint_{\mathcal{R}} K dA + \sum_{j=1}^{l} \theta_j = 2\pi\chi(\mathcal{R}),$$

where $\chi(\mathcal{R})$ is the Euler characteristic of $\mathcal{R}$.

Proof

Proof Let $\{T_i\}_{i=1}^{F}$ be a triangulation in the proposition. Let ι_{ik} be the interior angles of T_i. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g ds + \iint_{\mathcal{R}} K dA = \sum_{i=1}^{F} \sum_{k=1}^{3} \iota_{ik} - \pi F.$$

- E_1=number of external edges, E_2=number internal edges;

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- E_1=number of external edges, E_2=number internal edges;
- V_1=number of external vertices, V_2=internal vertices;

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- E_1=number of external edges, E_2=number internal edges;
- V_1=number of external vertices, V_2=internal vertices;
- W_1=number of external vertices which are not end points of and C_i, and W_2=number external vertices which are end points of C_i.

- $W_2 = l$, i.e. the number of exterior angles.
- $3F = 2E_2 + E_1,\ V_1 = E_1$.
- $$\sum_{i=1}^{F}\sum_{k=1}^{3}\iota_{ik} = 2\pi V_2 + \pi W_1 + \sum_{j=1}^{l}(\pi - \theta_j).$$

Hence

$$\sum_{i=1}^{F}\sum_{k=1}^{3}\iota_{ik} - \pi F = 2\pi V_2 + \pi W_1 + \sum_{j=1}^{I}(\pi - \theta_j) - \pi F.$$

$$\begin{aligned}\sum_{i=1}^{F}\sum_{k=1}^{3}\iota_{ik} - \pi F + \sum_{j=1}^{I}\theta_j &= 2\pi V_2 + \pi W_1 + \pi I - \pi F \\ &= 2\pi V_2 + \pi W_1 + \pi W_2 - \pi F \\ &= 2\pi V_2 + \pi V_1 - \pi F \\ &= 2\pi V - \pi V_1 - \pi F \\ &= 2\pi(F + V) - 3\pi F - \pi V_1 \\ &= 2\pi(F + V) - 2\pi E_2 - \pi E_1 - \pi E_1 \\ &= 2\pi(V - E + F).\end{aligned}$$

Corollary

Let M be a regular orientable compact surface. Then

$$\iint_M K dA = 2\pi\chi(M) = 4\pi(1-g),$$

where g is the genus of M. Hence: (i) $\int_M K dA > 0$ if and only if M is diffeomorphic to $\mathbb{S}^2$; (ii) $\int_M K dA = 0$ if and only if M is diffeomorphic to the torus; and (iii) $\int_M K dA < 0$ if and only if M is diffeomorphic to $\mathbb{S}^2$ with g handles attached for some $g \geq 2$.

Proposition

The Euler characteristic is well-defined (for piecewise smooth triangulation).

Proposition

Let T be a geodesic triangle on a regular surface. Then

$$\iint_T KdA = -\pi + (\iota_1 + \iota_2 + \iota_3)$$

where ι_i's are the interior angle. (RHS is called the excess of the triangle T).

A theorem of Jacobi

Theorem (Jacobi)

Let $\alpha : [0, l] \to \mathbb{R}^3$ be a regular smooth closed curve in $\mathbb{R}^3$ with nonzero curvature parametrized by arc length. Let $\mathbf{N}$ be the unit normal of α, which can be considered as a map from $[0, l]$ to $\mathbb{S}^2$. Assume that $\mathbf{N}$ is simple. Then $\mathbf{N}$ divides $\mathbb{S}^2$ in regions with equal area.

Theorem

Let α be a simple closed regular curve in $\mathbb{R}^2$. Then its interior is homeomorphic to an open disk.

One can first prove that the region is simply connected. Then use the Riemann mapping theorem.

Proof of Jacobi's Theorem: **N** is smooth simple closed. Hence it divides $\mathbb{S}^2$ in two regions, each is homeomorphic to an open disk in $\mathbb{R}^2$. Let $\mathcal{R}$ be one of the region. Then by Gauss-Bonnet, we have

$$2\pi = \iint_{\mathcal{R}} K dA + \int_{\partial\mathcal{R}} k_g ds = A(\mathfrak{R}) + \int_{\partial\mathcal{R}} k_g ds$$

where s is the arc length of $\partial\mathcal{R}$ which is $\mathbf{N}(\sigma)$ where σ is the arc-length of α and k_g is the geodesic curvature of $\mathbf{N}(\sigma)$.

Now $k_g = \langle \mathbf{N} \times \dot{\mathbf{N}}, \ddot{\mathbf{N}} \rangle$ where $\dot{\mathbf{N}} = \frac{d}{ds}\mathbf{N}$ etc. Let $\alpha' = \frac{d}{d\sigma}\alpha = \mathbf{T}$ etc.

$$\dot{\mathbf{N}} = \frac{d}{ds}\mathbf{N} = \frac{d\sigma}{ds}\frac{d}{d\sigma}\mathbf{N} = \frac{d\sigma}{ds}(-k\mathbf{T} - \tau\mathbf{B}).$$

$$\ddot{\mathbf{N}} = (\frac{d^2\sigma}{ds^2}(-k\mathbf{T} - \tau\mathbf{B}) + (-k'\mathbf{T} - \tau'\mathbf{B})(\frac{d\sigma}{ds})^2 - (k^2+\tau^2)\mathbf{N}(\frac{d\sigma}{ds})^2.$$

$$\mathbf{N} \times \dot{\mathbf{N}} = \frac{d\sigma}{ds}(k\mathbf{B} - \tau\mathbf{T})$$

So

$$k_g = (\frac{d\sigma}{ds})^3(-k\tau' + k'\tau).$$

Now $\frac{ds}{ds\sigma} = |\mathbf{N}'| = (k^2+\tau^2)^{\frac{1}{2}}$. Hence

$$k_g = \frac{d\sigma}{ds}\frac{-k\tau' + k'\tau}{k^2+\tau^2} = \frac{d\sigma}{ds}\frac{-\frac{\tau'}{k} + \frac{k'}{k^2}}{1 + \frac{\tau^2}{k^2}} = -\frac{d\sigma}{ds}\frac{(\frac{\tau}{k})'}{1 + \frac{\tau^2}{k^2}}.$$

Hence

$$\int_{\partial \mathcal{R}} k_g ds = -\int_{\partial \mathcal{R}} \frac{(\frac{\tau}{k})'}{1+\frac{\tau^2}{k^2}} d\sigma = 0.$$

The result follows.

Digression: Beltrami equations

Theorem

Near every point in a regular surface, one can introduce an isothermal parametrization.

Explain: Let the original first fundamental form be g_{ij} with coordinates x, y. Then isothermal coordinates be u, v. We want to find $\lambda > 0$, u, v so that

$$g_{11}(x')^2 + 2g_{12}x'y' + g_{22}(y')^2 = \lambda((u')^2 + (v')^2.$$

That is:

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$= \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

So we need:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

So

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

So we have:

$$u_x(av_x + bv_y) + u_y(bv_x + cv_y) = 0.$$

And $u_x = \rho(bv_x + cv_y), u_y = -\rho(av_x + bv_y)$. Now

$$au_x^2 + 2bu_xu_y + cu_y^2 = av_x^2 + 2bv_xv_y + cv_y^2.$$

$$\rho^2(ac - b^2) = 1.$$

Hence it remains to solve the Beltrami Equation

$$\begin{cases} u_x = \frac{bv_x + cv_y}{(ac-b^2)^{\frac{1}{2}}} \\ u_y = -\frac{av_x + bv_y}{(ac-b^2)^{\frac{1}{2}}} \end{cases}$$

Let $w = u + \mathbf{i}v, z = x + \mathbf{i}y$, the equation is equivalent to

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z}$$

where $\mu = (c - a - 2\mathbf{i}b)/(c + a + 2\sqrt{ac - b^2})$. Note that $|\nu| < 1$.
See Courant-Hilbert: Methods of mathematical physics, vol. 2.