Isometry

Definition

Let $F : M_1 \to M_2$ be a diffeomorphism. F is said to be an isometry if for any $p \in M_1$ and $q = F(p)$, the linear map $dF : M_1 \to M_2$ is an isometry as inner product spaces. If there is an isometry from M_1 onto M_2, then M_1 is said to be isometric to M_2.

Examples

- Let M_1 be the xy-plane parametrized by $\mathbf{X}(u, v) = (u, v, 0)$. Let M_2 be the circular cylinder parametrized by $\mathbf{Y}(u, v) = (\cos u, \sin u, v)$.

Examples

- Let M_1 be the xy-plane parametrized by $\mathbf{X}(u, v) = (u, v, 0)$. Let M_2 be the circular cylinder parametrized by $\mathbf{Y}(u, v) = (\cos u, \sin u, v)$.
- Consider the map $F : M_1 \to M_2$ so that $\mathbf{X}(u, v)$ is mapped into $\mathbf{Y}(u, v)$. This is not a diffeomorpism, but is a local diffeomorphism. Note that

$$dF(\mathbf{X}_u) = \mathbf{Y}_u, dF(\mathbf{X}_v) = \mathbf{Y}_v.$$

Moreover, $\langle \mathbf{X}_u, \mathbf{X}_u \rangle = 1 = \langle \mathbf{Y}_u, \mathbf{Y}_u \rangle$, $\langle \mathbf{X}_v, \mathbf{X}_v \rangle = 1 = \langle \mathbf{Y}_v, \mathbf{Y}_v \rangle$, $\langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0 = \langle \mathbf{Y}_u, \mathbf{Y}_u \rangle$. So this is a local isometry.

Let M_1 be the xy-plane with the negative axis deleted, parametrized by $\mathbf{X}(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta, 0)$. Let M_2 be the cone $\{z = k\sqrt{x^2 + y^2}\}$, so that $\cot \alpha = k$, $0 < 2\alpha < \pi$ is the angle at the vertex. Parametrize the cone by

$$\mathbf{Y}(\rho, \theta) = (\rho \sin \alpha \cos(\frac{\theta}{\sin \alpha}), \rho \sin \alpha \sin(\frac{\theta}{\sin \alpha}), \rho \cos \theta)$$

Then it is a local isometry.

Examples

Let M_1 be the catenoid parametrized by

$$\mathbf{X}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$$

Let M_2 be the helicoid given by

$$\mathbf{Y}(s, t) = (t \cos s, t \sin s, as).$$

Define a map F from M_1 to M_2 so that $(u, v) \to (s, t) = (u, a \sinh v)$. The Jacobian matrix is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & a \cosh v \end{pmatrix}$$

Then $dF(\mathbf{X}_u) = \mathbf{Y}_s, dF(\mathbf{X}_v) = a \cosh v \mathbf{Y}_t$. So

$$\langle \mathbf{X}_u, \mathbf{X}_u \rangle = a^2 \cosh^2 v = \langle dF(\mathbf{X}_u), dF(\mathbf{X}_u) \rangle$$

etc

Theorema Egregium of Gauss

Theorem

(Theorema Egregium of Gauss) The Guassian curvature K is invariant under isometries. That is to say, the Gaussian curvature depends only on the first fundamental form.

First proof

Recall the following.

- Let $A = (a_{ij})$, $B = (b_{ij})$ be two 3×3 matrices. Let $\mathbf{a}_i$ be the row vectors of A and $\mathbf{b}_j$ be the column vectors of B. Then

$$AB = (\langle \mathbf{a}_i, \mathbf{b}_i \rangle) .$$

First proof

Proof:

- Let $\mathbf{X}(u^1, u^2)$ be a local parametrization of a regular surface, and let g_{ij} be the coefficients of the first fundamental form and h_{ij} be the second fundamental form.

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- In the following, if **a**, **b**, **c** are three vectors, $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the ordered triple product of the three vectors. This is just equal to $\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$ as row vectors or as column vectors.

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- In the following, if **a**, **b**, **c** are three vectors, $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the ordered triple product of the three vectors. This is just equal to $\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$ as row vectors or as column vectors.
- $$\begin{aligned} h_{ij} =& \langle \mathbf{N}, \mathbf{X}_{ij} \rangle = \frac{(\mathbf{X}_{ij}, \mathbf{X}_1, \mathbf{X}_2)}{\sqrt{\det(g_{ij})}} \\ =:& \frac{\Theta_{ij}}{\sqrt{\det(g_{ij})}}. \end{aligned}$$

First proof

Proof:

- Let $\mathbf{X}(u^1, u^2)$ be a local parametrization of a regular surface, and let g_{ij} be the coefficients of the first fundamental form and h_{ij} be the second fundamental form.
- In the following, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors, $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the ordered triple product of the three vectors. This is just equal to $\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$ as row vectors or as column vectors.
- $$h_{ij} = \langle \mathbf{N}, \mathbf{X}_{ij} \rangle = \frac{(\mathbf{X}_{ij}, \mathbf{X}_1, \mathbf{X}_2)}{\sqrt{\det(g_{ij})}} =: \frac{\Theta_{ij}}{\sqrt{\det(g_{ij})}}.$$
- Now

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \det(g_{ij})^{-2} \left(\Theta_{11}\Theta_{22} - \Theta_{12}^2\right)$$

First proof, cont.

$$\Theta_{11}\Theta_{22} = \det\begin{pmatrix} \langle \mathbf{X}_{11}, \mathbf{X}_{22}\rangle & \langle \mathbf{X}_{11}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{11}, \mathbf{X}_{2}\rangle \\ \langle \mathbf{X}_{1}, \mathbf{X}_{22}\rangle & \langle \mathbf{X}_{1}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{1}, \mathbf{X}_{2}\rangle \\ \langle \mathbf{X}_{2}, \mathbf{X}_{22}\rangle & \langle \mathbf{X}_{2}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{2}, \mathbf{X}_{2}\rangle \end{pmatrix}$$

$$= \det\begin{pmatrix} \langle \mathbf{X}_{11}, \mathbf{X}_{22}\rangle & \frac{1}{2}(g_{11})_1 & (g_{12})_1 - \frac{1}{2}(g_{11})_2 \\ (g_{12})_2 - \frac{1}{2}(g_{22})_1 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{12} & g_{22} \end{pmatrix}$$

$$\Theta_{12}^2 = \det\begin{pmatrix} \langle \mathbf{X}_{12}, \mathbf{X}_{12}\rangle & \langle \mathbf{X}_{12}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{12}, \mathbf{X}_{2}\rangle \\ \langle \mathbf{X}_{1}, \mathbf{X}_{12}\rangle & \langle \mathbf{X}_{1}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{1}, \mathbf{X}_{2}\rangle \\ \langle \mathbf{X}_{2}, \mathbf{X}_{12}\rangle & \langle \mathbf{X}_{2}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{2}, \mathbf{X}_{2}\rangle \end{pmatrix}$$

$$= \det\begin{pmatrix} \langle \mathbf{X}_{12}, \mathbf{X}_{12}\rangle & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{12} & g_{22} \end{pmatrix}$$

First proof, cont.

Hence

$$\Theta_{11}\Theta_{22}-\Theta_{12}^2$$
$$=\det\begin{pmatrix} \langle \mathbf{X}_{11},\mathbf{X}_{22}\rangle-\langle \mathbf{X}_{12},\mathbf{X}_{12}\rangle & \frac{1}{2}(g_{11})_1 & (g_{12})_1-\frac{1}{2}(g_{11})_2 \\ (g_{12})_2-\frac{1}{2}(g_{22})_1 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{12} & g_{22} \end{pmatrix}$$
$$-\det\begin{pmatrix} 0 & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{12} & g_{22} \end{pmatrix}.$$

First proof, cont.

Now

$$
\begin{aligned}
\langle \mathbf{X}_{11}, \mathbf{X}_{22} \rangle - \langle \mathbf{X}_{12}, \mathbf{X}_{12} \rangle
&= \langle \mathbf{X}_{1}, \mathbf{X}_{22} \rangle_1 - \langle \mathbf{X}_{1}, \mathbf{X}_{221} \rangle - \langle \mathbf{X}_{1}, \mathbf{X}_{12} \rangle_2 + \langle \mathbf{X}_{1}, \mathbf{X}_{122} \rangle \\
&= \left((g_{12})_2 - \frac{1}{2}(g_{22})_1 \right)_1 - \frac{1}{2} g_{11,22} \\
&= g_{12,12} - \frac{1}{2}(g_{11,22} + g_{22,11}).
\end{aligned}
$$

First proof, cont.

Hence

$$
\begin{aligned}
&(\det(g_{ij}))^2 K \\
=&\det\begin{pmatrix} g_{12,12}-\frac{1}{2}(g_{11,22}+g_{22,11}) & \frac{1}{2}(g_{11})_1 & (g_{12})_1-\frac{1}{2}(g_{11})_2 \\ (g_{12})_2-\frac{1}{2}(g_{22})_1 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{12} & g_{22} \end{pmatrix} \\
&-\det\begin{pmatrix} 0 & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{12} & g_{22} \end{pmatrix}.
\end{aligned}
$$

Hence K depends only on g_{ij} and their derivatives up to second order.

Christoffel symbols

Let $\mathbf{X}(u^1, u^2)$ is a coordinate parametrization. Let $\mathbf{X}_i = \mathbf{X}_{u^i}$, $g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle$, $(g^{ij}) = (g_{ij})^{-1}$. Then

$$\mathbf{X}_{ij} = \Gamma_{ij}^k \mathbf{X}_k + h_{ij} \mathbf{N}. \tag{1}$$

(Einstein summation convention: repeated indices mean summation.)

Γ_{ij}^k are called the Christoffel symbols for this parametrization.

To compute Γ_{ij}^k

Lemma

$\Gamma_{ij}^k = \Gamma_{ji}^k$ *and*

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right).$$

where $g_{ij,l} = \frac{\partial}{\partial u^l} g_{ij}$ *etc.*

Proof: $\mathbf{X}_{ij} = \mathbf{X}_{ji}$, so $\Gamma_{ij}^k = \Gamma_{ji}^k$.

$$\langle \mathbf{X}_{ij}, \mathbf{X}_l \rangle = \Gamma_{ij}^k g_{kl}$$

So

$$g_{il,j} - \langle \mathbf{X}_i, \mathbf{X}_{lj} \rangle = \Gamma_{ij}^k g_{kl}$$

So

$$g_{il,j} = \Gamma_{ij}^k g_{kl} + \Gamma_{lj}^k g_{ki}.$$

Proof, cont.

Hence we have

$$\begin{cases} g_{il,j} = & \Gamma^k_{ij} g_{kl} + \underline{\Gamma^k_{lj} g_{ki}}. \\ g_{jl,i} = & \Gamma^k_{ji} g_{kl} + \underline{\underline{\Gamma^k_{li} g_{kj}}}. \\ g_{ij,l} = & \underline{\underline{\Gamma^k_{il} g_{kj}}} + \underline{\Gamma^k_{jl} g_{ki}}. \end{cases}$$

Hence

$$g_{il,j} + g_{jl,i} - g_{ij,l} = 2\Gamma^k_{ij} g_{kl}.$$

From this the result follows.

Examples

- Let M be the xy-plane parametrized by $\mathbf{X}(u, v) = (u, v, 0)$. Then $\Gamma_{ij}^{k} = 0$ for all i, j, k.

So $\Gamma_{22}^{1} = -r$, $\Gamma_{12}^{2} = r^{-1}$, all other Γ's are zero.

Examples

- Let M be the xy-plane parametrized by $\mathbf{X}(u, v) = (u, v, 0)$. Then $\Gamma_{ij}^k = 0$ for all i, j, k.
- If we use polar coordinates, $\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, 0)$. If $u^1 \leftrightarrow r, u^2 \leftrightarrow \theta$. Then $g_{11} = 1, g_{12} = 0, g_{22} = r^2$. So $g^{11} = 1, g^{12} = 0, g^{22} = r^{-2}$. Then

$$\Gamma_{ij}^1 = \frac{1}{2} g^{1k} \left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) = \frac{1}{2} \left(g_{i1,j} + g_{j1,i} - g_{ij,1}\right)$$

Similarly,

$$\Gamma_{ij}^2 = \frac{1}{2} g^{2k} \left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) = \frac{1}{2} r^{-2} \left(g_{i2,j} + g_{j2,i} - g_{ij,2}\right).$$

So $\Gamma_{22}^1 = -r$, $\Gamma_{12}^2 = r^{-1}$, all other Γ's are zero.

Examples, cont.

Consider the surface of revolution given by

$$\mathbf{X}(u,v) = (\alpha(v)\cos u, \alpha(v)\sin u, \beta(v))$$

with $\alpha > 0$. Consider $u^1 \leftrightarrow u, u^2 \leftrightarrow v$. Then
$g_{11} = \alpha^2, g_{12} = 0, g_{22} = (\alpha')^2 + (\beta')^2$. So
$g^{11} = \alpha^{-2}, g^{12} = 0, g^{22} = \left((\alpha')^2 + (\beta')^2\right)^{-1}$.

$$\Gamma^1_{ij} = \frac{1}{2} g^{1k}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) = \frac{1}{2}\alpha^{-2}\left(g_{i1,j} + g_{j1,i} - g_{ij,1}\right).$$

So

$$\Gamma^1_{11} = \frac{1}{2}\alpha^{-2} g_{11,1} = 0, \; \Gamma^1_{22} = \frac{1}{2}\alpha^{-2} g_{22,1} = 0,$$

$$\Gamma^1_{12} = \frac{1}{2}\alpha^{-2} g_{11,2} = \frac{\alpha'}{\alpha}.$$

Examples, cont.

Similarly,

$$\Gamma_{ij}^{2} = \frac{1}{2} g^{2k} \left(g_{ik,j} + g_{jk,i} - g_{ij,k} \right) = \frac{1}{2} g^{22} \left(g_{i2,j} + g_{j2,i} - g_{ij,2} \right).$$

Hence

$$\Gamma_{11}^{2} = -\frac{1}{2} g^{22} g_{11,2} = -\frac{\alpha\alpha'}{(\alpha')^2 + (\beta')^2}, \Gamma_{22}^{2} = \frac{1}{2} g^{22} g_{22,2} = \frac{\alpha'\alpha'' + \beta'\beta''}{(\alpha')^2 + (\beta')^2}.$$

$$\Gamma_{12}^{2} = \frac{1}{2} g^{22} g_{22,1} = 0.$$

Examples, cont.

In general, if $g_{12} = 0$, then

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right) = \frac{1}{2}g^{kk}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right)$$

no summation. So

$$\Gamma^1_{11} = \frac{1}{2}g^{11}g_{11,1}, \ \Gamma^2_{11} = -\frac{1}{2}g^{22}g_{11,2};$$

$$\Gamma^1_{22} = -\frac{1}{2}g^{11}g_{22,1}, \quad \Gamma^2_{22} = \frac{1}{2}g^{22}g_{22,2};$$

$$\Gamma^1_{12} = \frac{1}{2}g^{11}g_{11,2}, \quad \Gamma^2_{12} = \frac{1}{2}g^{22}g_{22,1}.$$

Second proof of Theorema Egregium of Gauss

Theorem

With the above notations, then

$$2K = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{lk}\Gamma^l_{ji} - \Gamma^k_{lj}\Gamma^l_{ki}\right) = g^{ij}(\Gamma^k_{i[j,k]} + \Gamma^k_{l[k}\Gamma^l_{j]i}).$$

Here $T_{[ij]k} = T_{ijk} - T_{jik}$ *etc.*

Compare with higher dimensional Riemannian curvature:

$$R^l_{ijk} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{js}\Gamma^s_{ik} - \Gamma^l_{ks}\Gamma^s_{ij}$$

Proof

Proof: Let $\mathcal{S}$ be the shape operator, then

$$-\mathbf{N}_i = \mathcal{S}(\mathbf{X}_i) = a_i^j \mathbf{X}_j.$$

$$\begin{aligned}\mathbf{X}_{ijm} =& h_{ij,m}\mathbf{N} + h_{ij}\mathbf{N}_m + \Gamma_{ij,m}^k \mathbf{X}_k + \Gamma_{ij}^k \mathbf{X}_{km} \\ &= \left(h_{ij,m} + \Gamma_{ij}^k h_{km}\right)\mathbf{N} + \left(-h_{ij} a_m^k + \Gamma_{ij,m}^k + \Gamma_{ij}^s \Gamma_{sm}^k\right)\mathbf{X}_k\end{aligned}$$

Since $\mathbf{X}_{ijm} = \mathbf{X}_{imj}$, we have

$$\left(-h_{ij} a_m^k + \Gamma_{ij,m}^k + \Gamma_{ij}^s \Gamma_{sm}^k\right)\mathbf{X}_k = \left(-h_{im} a_j^k + \Gamma_{im,j}^k + \Gamma_{im}^s \Gamma_{sj}^k\right)\mathbf{X}_k$$

Or

$$h_{ij} a_m^k - h_{im} a_j^k = \Gamma_{ij,m}^k - \Gamma_{im,j}^k + \Gamma_{ij}^s \Gamma_{ms}^k - \Gamma_{im}^s \Gamma_{js}^k$$

Proof, cont.

Now the matrix of the shape operator is:

$$(a_i^j) = (h_{ij})(g_{ij})^{-1}$$

So $h_{ji} = h_{ij} = a_i^l g_{lj} = a_j^l g_{li}$. Hence

$$a_i^l g_{lj} a_m^k - a_m^l g_{li} a_j^k = \Gamma_{ij,m}^k - \Gamma_{im,j}^k + \Gamma_{ij}^s \Gamma_{ms}^k - \Gamma_{im}^s \Gamma_{js}^k.$$

Let $m = k$ and sum on k

$$\begin{aligned} & g^{ij} \left(\Gamma_{ij,k}^k - \Gamma_{ik,j}^k + \Gamma_{ij}^s \Gamma_{ks}^k - \Gamma_{ik}^s \Gamma_{js}^k \Gamma_{ij,k} \right) \\ =& g^{ij} \left(a_i^l g_{lj} a_k^k - a_k^l g_{li} a_j^k \right) \\ =& (\sum_i a_i^i)^2 - \sum_{l,k} a_l^k a_k^l \\ =& 2a_{11} a_{22} - 2a_1^2 a_2^1 \\ =& 2K \end{aligned}$$

Compatibility conditions

Given (g_{ij}) which is symmetric and positive definite and (h_{ij}) which is symmetric, can we find $\mathbf{X}(u^1, u^2)$ so that the first fundamental form is h_{ij}? If $\mathbf{X}_i$ exist, then we can find $\mathbf{X}$. The restriction on $\mathbf{X}_i$ are

$$\mathbf{X}_{ijk} = \mathbf{X}_{ikj}, \quad \mathbf{N}_{ij} = \mathbf{N}_{ji}.$$

Hence we have

$$\left(-h_{ij}a_m^k + \Gamma_{ij,m}^k + \Gamma_{ij}^s\Gamma_{sm}^k\right)\mathbf{X}_k = \left(-h_{im}a_j^k + \Gamma_{im,j}^k + \Gamma_{im}^s\Gamma_{sj}^k\right)\mathbf{X}_k$$

with $a_i^j = h_{il}g^{lj}$. We have three relations for each $\mathbf{X}_i$. Now

$$\begin{aligned} -\mathbf{N}_{ij} =& (a_i^k\mathbf{X}_k)_j \\ =& (a_i^k)_j\mathbf{X}_k + a_i^k(\Gamma_{jk}^l\mathbf{X}_l + a_i^k h_{jk}\mathbf{N}) \\ =& \left((a_i^k)_j + a_i^l\Gamma_{jl}^k\right)\mathbf{X}_k + a_i^k h_{jk}\mathbf{N} \end{aligned}$$

So we also need, for $k = 1, 2$

$$(a_i^k)_j + a_i^l \Gamma_{jl}^k = (a_j^k)_i + a_j^l \Gamma_{il}^k.$$

These are called Gauss equations and Mainardi-Codazzi equations respectively.