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■ A (*parametrized smooth*) *curve* $\alpha(t)$ is a smooth map

$$\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$$

from an interval I in $\mathbb{R}$ into $\mathbb{R}^3$ so that α is smooth. α is said to be *regular* if $\alpha' \neq 0$.

■ Let $\alpha : (a, b) \to \mathbb{R}^3$ is a curve. Let $f : (c, d) \to (a, b)$ with $t = f(\sigma)$ such that $f' > 0$, then $\alpha(f(\sigma)) : (c, d) \to \mathbb{R}^3$ is said to be a *reparametrization* of α.

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Arc-length

■ Let α be a regular curve defined on $[a, b]$ and let $t_0 \in [a, b]$, the *arc-length* is defined as:

$$s(t) = \int_{t_0}^{t} |\alpha'(u)|du.$$

■ If $s(a) = -L_1, s(b) = L_2$, then $\alpha(s) = \alpha(s(t))$ is a reparametrization of α and $\alpha(s)$ is said to be parametrized by arc-length.

■ $\alpha = \alpha(t)$ is parametrized by arc-length, that is t 'represents' arc-length from a fixed point iff $|\alpha'| = 1$.

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The Frenet formula Let $\alpha(s)$ be the regular curve parametrized by arc length.
Let $\vec{T} = \alpha'$ (tangent). Then

$$k(s) := |T'|(s) \quad \text{(curvature)};$$
$$N(s) := \frac{1}{k(s)} T'(s) \quad \text{(normal, if } k > 0\text{)};$$
$$B(s) := T(s) \times N(s) \quad \text{(binormal, if } k > 0\text{)}.$$

Fact: $B' = -\tau N$, τ is called the torsion of α.

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Theorem

(Frenet formula) Let α be a regular curve with curvature $k > 0$. Then

$$\left(\begin{array}{c} T \\ N \\ B \end{array} \right)' = \left(\begin{array}{ccc} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{array} \right) \left(\begin{array}{c} T \\ N \\ B \end{array} \right).$$

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We summarize some properties on curves:

Let α be a regular curves in $\mathbb{R}^3$ parametrized by arc length.

(i) Suppose the curvature $k \equiv 0$ if and only if α is a straight line.

(ii) Suppose the curvature $k > 0$ and the torsion $\tau \equiv 0$ if and only if α is a plane curve.

(iii) Suppose the curvature $k = k_0 > 0$ is a constant and $\tau \equiv 0$, then α is a circular arc with radius $1/k_0$.

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(iv) Suppose the curvature $k > 0$ and the torsion $\tau \neq 0$ everywhere. α lies on s sphere if and only if $\rho^2 + (\rho')^2 \lambda^2 =$ constant, where $\rho = 1/k$ and $\lambda = 1/\tau$.

(v) Suppose the curvature $k = k_0 > 0$ is a constant and $\tau = \tau_0$ is a constant. Then α is a circular helix.

(vi) Suppose α is defined on $[a, b]$. Let $\mathbf{p} = \alpha(a)$ and $\mathbf{q} = \alpha(b)$. Then $L(\alpha) \leq |\mathbf{p} - \mathbf{q}|$. Equality holds if and only if α is the straight line from $\mathbf{p}$ to $\mathbf{q}$.

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Ref: *Ordinary differential equations, Birkoff and Rota*

We only consider the special case of linear ODE. Let $A(t) = (a_{ij}(t))_{n \times n}$ be a smooth family $n \times n$ matrix, $t \in [a, b]$. Consider the following initial valued problem (IVP): Given A and a constant $\mathbf{x}_0 \in \mathbb{R}^n$, to find $\mathbf{x} : [a, b] \to \mathbb{R}^n$ satisfying:

$$\begin{cases} \mathbf{x}'(t) & = A(t)\mathbf{x}(t), \quad t \in [a, b]; \\ \mathbf{x}(a) & = \mathbf{x}_0. \end{cases}$$

Theorem

Given any $\mathbf{x}_0 \in \mathbb{R}^n$, the exists a unique solution of the above IVP.

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[Proof](Sketch) For simplicity let us assume $a = 0$.
Existence: Define inductively, with $\mathbf{x}_0(t) = \mathbf{x}_0$ for all t, and

$$\mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_0^t A(\tau)\mathbf{x}_k(\tau)d\tau.$$

for $k \geq 0$. Let $M = \sup_{t \in [a,b]} ||A||(t)$ and $||A(t)||^2 = \text{tr}(AA^T(t))$.
For $k \geq 1$, we have

$$|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq M \int_0^t |\mathbf{x}_k(\tau) - \mathbf{x}_{k-1}(\tau)|d\tau.$$

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Inductively, we have (why?)

$$
\begin{aligned}
&|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \\
\leq &M^k \int_0^t \int_0^{\tau_{k-1}} \dots \int_0^{\tau_2} \int_0^{\tau_1} |\mathbf{x}_1(\tau_1) - \mathbf{x}_0(\tau_1)| d\tau_1 d\tau_2 \dots d\tau_{k-1} d\tau_k \\
\leq &\frac{M^k b^k S}{k!}
\end{aligned}
$$

where integration is over the domain $t \geq \tau_k \geq \dots \geq \tau_1$ and $S = \sup_{t \in [0,b]} |\mathbf{x}_1(t) - \mathbf{x}_0(t)|$.
Hence $\sum_{k=1}^{\infty} |\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq C$ for some constant C for all $t \in [0, b]$. This implies that $\mathbf{x}_k \to \mathbf{x}_\infty$ uniformly on $[0, b]$ which satisfies:

$$
\mathbf{x}_\infty(t) = \mathbf{x}_0 + \int_0^t A(\tau)\mathbf{x}_\infty(\tau)d\tau,
$$

(why?) Now $\mathbf{x}_\infty$ is the solution of the above IVP.

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Theorem

Let $\kappa(s) > 0$ and $\tau(s)$ be smooth function on (a, b). There exists a regular curve $\alpha : (a, b) \to \mathbb{R}^3$ with $|\alpha'| = 1$, such that the curvature and torsion of α are k, τ respectively.

Moreover, α is unique in the sense that if β is another curve satisfying the above conditions, then $\beta(s) = \alpha(s)P + \vec{c}$ for some constant orthogonal matrix P and some constant vector $\vec{c}$. **Here α, β are considered as row vectors.**

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Proof.

Uniquess: Sufficient to prove that if $\mathbf{x}_0 = \mathbf{0}$, then any solution must be trivial. So let $\mathbf{x}$ be such a solution, then

$$\frac{d}{dt}||\mathbf{x}||^2 = 2\langle A\mathbf{x}, \mathbf{x}\rangle \leq 2M||\mathbf{x}||^2.$$

Hence

$$\frac{d}{dt}\left(\exp(-2Mt)||\mathbf{x}||^2\right) \leq 0.$$

This will imply that $||\mathbf{x}||^2 \equiv 0$. (Why?) $\qquad\square$

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(**Existence**): Let

$$A(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}.$$

Let $X(s)$ be the 3×3 matrix and fix s_0 which is the solution of:

$$\begin{cases} X' & = AX \quad \text{in } (a, b); \\ X(s_0) & = I. \end{cases}$$

The solution exists by a theorem in ODE. We claim that X is orthogonal with determinant 1.

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In fact

$$(X^tX)' = (X^t)'X + X^tX' = (AX)^tX + X^tAX = X^tA^tX + X^tAX = 0$$

because $A^t = -A$. Hence $X^tX = I$ because $X^t(s_0)X(s_0) = I$.
(**Using $(XX^t)'$ may be more involved.**) Hence $X(s)$ is
orthogonal. Since $\det X(s) = 1$ or -1 and initially, $\det X(s_0) = 1$,
we have $\det X(s) = 1$. Write

$$X = \begin{pmatrix} \widetilde{T} \\ \widetilde{N} \\ \widetilde{B} \end{pmatrix}.$$

Define $\alpha(s) = \int_{s_0}^{s} \widetilde{T}(\sigma)d\sigma$. Let T, N, B be the tangent, principal
normal and binormal of α, and let $\kappa_\alpha, \tau_\alpha$ be the curvature and
torsion of α.

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Then $\alpha' = \widetilde{T}$ which has length 1. So $T = \widetilde{T}$. Moreover,

$$\kappa_\alpha N = T' = \widetilde{T}' = \kappa \widetilde{N}.$$

we have $\kappa_\alpha = \kappa$ and $N = \widetilde{N}$. Since $\widetilde{T}, \widetilde{N}, \widetilde{B}$ are positively oriented, we conclude that

$$B = T \times N = \widetilde{T} \times \widetilde{N} = \widetilde{B},$$

and

$$-\tau_\alpha N = B' = \widetilde{B}' = -\tau \widetilde{N} = -\tau N.$$

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Uniqness: Let α, β as in the theorem. Let $T_\alpha, N_\alpha, B_\alpha$ be the unit tangent, principal normal, binormal of α; and let $T_\beta, N_\beta, B_\beta$ be the unit tangent, principal normal, binormal of β. Fix $s_0 \in (a, b)$. Let P be an orthogonal matrix with determinant 1 such that

$$\left(\begin{array}{c} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{array} \right) = \left(\begin{array}{c} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{array} \right) P.$$

Here $T_\alpha, \ldots$, etc are considered as row vectors. Let $\gamma(s) = \alpha(s)P$. Let $T_\gamma, N_\gamma, B_\gamma$ be unit tangent, principal normal, binormal of γ.

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Then

$$T_\gamma = \gamma' = \alpha' P = T_\alpha P,$$

$$\kappa N_\gamma = T'_\gamma = T'_\alpha P = \kappa NP.$$

and so $T_\gamma = T_\alpha P, N_\gamma = N_\alpha P$. Hence $B_\gamma = B_\alpha P$. We have

$$\left(\begin{array}{c} T_\gamma \\ N_\gamma \\ B_\gamma \end{array} \right)' = \left(\begin{array}{c} T_\alpha \\ N_\alpha \\ B_\alpha \end{array} \right)' P = A \left(\begin{array}{c} T_\alpha \\ N_\alpha \\ B_\alpha \end{array} \right) P = A \left(\begin{array}{c} T_\gamma \\ N_\gamma \\ B_\gamma \end{array} \right)$$

where A is as above. Since

$$\left(\begin{array}{c} T_\gamma(s_0) \\ N_\gamma(s_0) \\ B_\gamma(s_0) \end{array} \right) = \left(\begin{array}{c} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{array} \right) P = \left(\begin{array}{c} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{array} \right).$$

we have $T_\gamma = T_\beta$, by uniqueness theorem of ODE. So $\gamma(s) + \vec{c} = \beta(s)$ for some constant vector $\vec{c}$. That is: $\beta(s) = \alpha(s)P + \vec{c}$.

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Proposition

Let $\alpha(s)$ be a plane curve parametrized by arc length defined on (a, b). Let $s_0 \in (a, b)$. Suppose $\kappa(s_0) > 0$. Then the following are true:

(i) *For any $s_1 < s_2 < s_3$ sufficiently close to s_0, $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear.*

(ii) *For $s_1 < s_2 < s_3$ sufficiently close to s_0 so that $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear, let $c(s_1, s_2, s_3)$ be the center of the unique circle $C(s_1, s_2, s_3)$ passing through $\alpha(s_1), \alpha(s_2), \alpha(s_3)$. As $s_1, s_2, s_3 \to s_0$, $C(s_1, s_2, s_3)$ will converge to a circle passing through $\alpha(s_0)$ tangent to α at $\alpha(s_0)$ with radius $1/\kappa(s_0)$*

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[Proof] (i) Suppose $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ lie on a straight line. Then

$$\langle \alpha(s_i) - \vec{v}, \vec{n} \rangle = 0$$

for some constant vectors $\vec{v}, \vec{n}$ with $|\vec{n}| = 1$, for $i = 1, 2, 3$. Let $f(s) = \langle \alpha(s) - \vec{v}, \vec{n} \rangle$. Then $f(s_i) = 0$ for $i = 1, 2, 3$. Hence $f'(\xi_1) = f'(\xi_2) = 0$ for some $s_1 < \xi_1 < s_2 < \xi_2 < s_3$ and $f''(\eta) = 0$ for some $\xi_1 < \eta < \xi_2$. That is:

$$\begin{cases} \langle \alpha'(\xi_1), \vec{n} \rangle = & \langle \alpha'(\xi_2), \vec{n} \rangle = 0; \\ \langle \alpha''(\eta), \vec{n} \rangle = & 0. \end{cases}$$

As $s_1, s_2, s_3 \to s_0$, $\vec{n} \to N(s_0)$ and $\alpha''(\eta) = \kappa(s_0)N(s_0)$. This implies $\kappa(s_0) = 0$. Contradiction.

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(ii) Let $C(s_1, s_2, s_3)$ be given by

$$||\mathbf{x} - c|| = r.$$

where $c = c(s_1, s_2, s_3)$.
Let $h(s) = ||\alpha(s) - c||^2$. Then $h(s_i) = r^2$ for $i = 1, 2, 3$. Hence
$h'(\xi_1) = h'(\xi_2) = 0$ for some $s_1 < \xi_1 < s_2 < \xi_2 < s_3$ and $h''(\eta) = 0$
for some $\xi_1 < \eta < \xi_2$. Hence

$$\begin{cases} \langle \alpha'(\xi_1), \alpha(\xi_1) - c \rangle & = \langle \alpha'(\xi_2), \alpha(\xi_2) - c \rangle = 0; \\ \langle \alpha''(\eta), \alpha(\eta) - c \rangle + 1 & = 0. \end{cases}$$

If $c \to c_\infty$ for some sequence $s_1 < s_2 < s_3 \to s_0$, then

$$\langle \alpha'(s_0), \alpha(s_0) - c_\infty \rangle = 0, \quad \langle \alpha''(s_0), \alpha(s_0) - c_\infty \rangle = -1$$

So $c_\infty - \alpha(s_0) = \frac{1}{\kappa(s_0)} N(s_0)$. From this the result follows.
The limiting circle is called the *osculating circle*.

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Proposition

Let $\alpha(t)$ be a regular curve with nonzero curvature. Then the curvature and torsion are given by:

$$\begin{cases} \kappa = & \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \\ \tau = & \frac{<\alpha' \times \alpha'', \alpha'''>}{|\alpha' \times \alpha''|^2}. \end{cases}$$

Here $'$ always means differentiation with respect to t.

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Proof.

Let $\alpha(t)$ be a regular curve with nonzero curvature. Then

$$\alpha' = |\alpha'|T,$$

$$\alpha'' = \kappa|\alpha'|^2 N + |\alpha'|^{-1} < \alpha', \alpha'' > T. \qquad (1)$$

Hence

$$< \alpha'', \alpha'' > = \kappa^2|\alpha'|^4 + |\alpha'|^{-2} < \alpha', \alpha'' >^2,$$

and

$$\kappa^2 = \frac{< \alpha'', \alpha'' >< \alpha', \alpha' > - < \alpha', \alpha'' >^2}{|\alpha'|^6}$$

$$= \frac{|\alpha' \times \alpha''|^2}{|\alpha'|^6}.$$