Assignment 6, Due 11:59 pm, Monday Dec 7, 2020

(1) Consider a surface of revolution:

$$\mathbf{X}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

with $f > 0$ and $f_v^2 + g_v^2 > 0$. Suppose $\alpha(s)$ is a geodesic parametrized by arc length which is not a meridian. Show that $u' \neq 0$ and so $s = s(u)$ can be expressed as a function of u. Show also that

$$\frac{dv}{du} = \pm\frac{f}{c}\sqrt{\frac{f^2 - c^2}{f_v^2 + g_v^2}}$$

for some constant $c \neq 0$.

(2) Show that any geodesic of a paraboloid $z = x^2 + y^2$ which is not a meridian intersects itself an infinitely many times. (See Do Carmo: p.258–260).

(3) Prove that if M is a regular surface with nonpositive Gaussian curvature. Prove that two geodesics from a point p cannot meet again so that they form the boundary of a simple region, i.e. an open set which is homeomorphic to a disk.

(4) Let M be compact surface with positive Gaussian curvature. Prove that M is homeomorphic to a sphere. Prove also that if there exist two distinct simple closed geodesics on M, then they must intersect.

(5) Let M be a surface diffeomorphic to a cylinder. Suppose the Gaussian curvature of M is negative. Then M has at most one closed geodesic.

(6) Let M be a regular surface in $\mathbb{R}^3$ without boundary. Suppose the Gaussian curvature of M is positive. Let γ be a simple closed geodesic in M. Let A, B be the regions of M which have γ as a common boundary. Show that the Gauss images under the Gauss maps of A and B have the same area.