Assignment 5, Due 11:59 pm, Thursday Nov 19, 2020

(1) (a) Find the absolute value of the curvature of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points $(a, 0)$ and $(0, b)$. Assuming $a, b > 0$.

(b) Intersect the cylinder $C = \{(x, y, z) | x^2 + y^2 = 1\}$ with a plane passing through the x-axis and making an angle θ with the xy-plane. Show that the curve α is an ellipse. Also find the absolute value of the geodesic curvature of α at the points where α meets their axes (i.e. major and minor axes of the ellipse).

(2) Let $\alpha(\tau)$ be a regular curve on a regular surface M, where τ may not be proportional to arc length. Let $\alpha' = \frac{\partial \alpha}{\partial \tau}$, etc. Prove that after reparametrization, α is a geodesic if and only if $(\alpha'')^T = \lambda(\tau)\alpha'$ for some smooth function τ on α.

(3) Write down the differential equations for the geodesics on the torus:

$$\mathbf{X}(u, v) = ((a + r\cos v)\cos u, (a + r\cos v)\sin u, r\sin v)$$

with $a > r > 0$. Also, show that if α is a geodesic start at a point on the topmost parallel $(a\cos u, a\sin u, r)$ and is tangent to this parallel, then α will stay in the region with $-\pi/2 \le v \le \pi/2$.

Find also the geodesic curvature of the topmost parallel.

(4) Let $\mathbf{X} : U \to M$, $(u_1, u_2) \to \mathbf{X}(u_1, u_2)$, be a coordinate parametization, with U being an open set in $\mathbb{R}^2$. Suppose the first fundamental form in this coordinate satisfies $g_{12} = 0$.

(a) Let $g_{11} = E, g_{22} = G$. Find the equations of geodesic and find Γ_{ij}^k. in terms of E, G and their derivatives.

(b) Suppose $g_{11} = g_{22} = \exp(2f)$ for some smooth function f, Find Γ_{ij}^k in terms of f and its derivatives.

(5) With the same assumptions and notation as in the previous exercise, part (b). Let $\mathbf{e_1} = \mathbf{X}_1/|\mathbf{X}_1|$, $\mathbf{e_2} = \mathbf{X}_2/|\mathbf{X}_2|$, and $\mathbf{N} = \mathbf{e_1} \times \mathbf{e_2}$. Let $\alpha(s)$ be a geodesic on M such that $\alpha(s) = \mathbf{X}(u_1(s), u_2(s))$. Let $\theta(s)$ be a smooth function on s such that $\alpha'(s) = \mathbf{e_1}(s)\cos\theta(s) + \mathbf{e_2}(s)\sin\theta(s)$, where $\mathbf{e_i}(s) = \mathbf{e_i}(\alpha(s))$.

(a) Show that $\mathbf{a} := \mathbf{N} \times \alpha' = -\mathbf{e_1}(s)\sin\theta(s) + \mathbf{e_2}(s)\cos\theta(s)$.

(b) Show also that

$$\begin{aligned} k_g &= -\langle \alpha', \mathbf{a}' \rangle \\ &= \exp(-2f)\langle \frac{d}{ds}\mathbf{X}_1, \mathbf{X}_2 \rangle + \theta' \\ &= \left(-u' \frac{\partial f}{\partial v} + v' \frac{\partial f}{\partial u} \right) + \theta'. \end{aligned}$$

(Note that if $f = 1$, i.e. M is a plane, then $k_g = \theta'$.)