**Assignment 4, Due 11:59 pm, Thursday Nov 5, 2020**

(1) Show that the helicoid:

$$\mathbf{X}(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au),$$

and the Enneper's surface

$$\mathbf{X}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$

are minimal surfaces.

(2) A surface of revolution $M$ parametrized by $\mathbf{X}(u,v) = (u, h(u)\cos v, h(u)\sin v)$ has nonzero constant mean curvature if and only if the function $h(u)$ satisfies

$$h^2 \pm \frac{2ah}{\sqrt{1+(h')^2}} = b.$$

where $a > 0$ and $b$ are constants.

(Hint: Suppose $H = -1/(2a)$ with $a > 0$. Show that

$$\frac{d}{du}\left(h^2 + \frac{2ah}{\sqrt{1+(h')^2}}\right) = 0.)$$

(3) Prove that if $\mathbf{X}$ is an orthogonal parametrization, i.e. $F = 0$, then the Gaussian curvature is given by:

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

Suppose in addition $E = G$ everywhere, then

$$K = -e^{-2f}\Delta f$$

where $f$ is such that $E = e^{2f}$ (i.e. $f = \frac{1}{2}\log E$), and $\Delta$ is the Laplacian operator:

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}.$$

(4) Verify that the surfaces:

$$\mathbf{X}(u,v) = (u\cos v, u\sin v, \log u)$$

and

$$\mathbf{Y}(u,v) = (u\cos v, u\sin v, v)$$

have equal Gaussian curvature at that points $\mathbf{X}(u,v), \mathbf{Y}(u,v)$ but the coefficients of the first fundamental forms at points $\mathbf{X}(u,v), \mathbf{Y}(u,v)$ are not the same.

(5) Suppose a regular surface $M$ is parametrized by $u^1, u^2$ so that the first fundamental form is given by

$$g_{11} = g_{22} = \frac{4}{1 - \sum_{i=1}^{2} (u^i)^2}, \quad g_{12} = 0.$$

Find the Gaussian curvature of the surface. Here we assume that $(u^1)^2 + (u^2)^2 < 1$.

(Remark: Compare the coefficentes of the first fundamental form of the sphere in stereographic projection in Problem 5, assignment 2.)