Assignment 3, Due 16/10/2020 (Friday) on or before 11:59 pm

(1) Let M_1, M_2 be two regular surfaces and let $F : M_1 \to M_2$ be a smooth map. Let $p \in M$ and $q = f(p) \in M_2$. Define the differential $dF : T_p(M_1) \to T_q(M_2)$ as follows: For any $\mathbf{v} \in T_p(M)$, let $\alpha(t)$ be a smooth curve on M_1 so that $\alpha(0) = p, \alpha'(0) = \mathbf{v}$. Define

$$dF_p(\mathbf{v}) = \frac{d}{dt}F(\alpha(t))|_{t=0}.$$

Prove that dF_p is well-defined and is linear.

(Consider the Gauss map $\mathbf{N} : M \to \mathbb{S}^2$. One can identify $T_p(M)$ with $T_{\mathbf{N}(p)}(\mathbb{S}^2)$. Then $\mathcal{S}_p = -d\mathbf{N}_p$.)

(2) Consider the surface of revolution by rotating the curve $\alpha(v) = (\phi(v), 0, \psi(v))$ on the xz plane about the z-axis: $\mathbf{X}(u, v) = (f(v)\cos u, f(v)\sin u, \psi(v))$. Assuming v is the arc length of α. Find the coefficients of the first and second fundamental forms of the surface with respect to this parametrization. Find also the Gaussian curvature and mean curvature.

(3) Consider the tractrix Let $\alpha : (0, \frac{\pi}{2}) \to xz$-plane given by

$$\alpha(t) = \left(\sin\, t, 0, \cos\, t + \log\, \tan\frac{t}{2}\right).$$

Show that the Gaussian curvature of the surface of revolution obtained by rotating α about the z-axis is -1. The surface is called the pseudosphere.

(4) Let $M = \{(x, y, z)|\ z = x^2 + ky^2\}$, with $k > 0$. Show that $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ form a basis of $T_p(M)$ where $p = (0, 0, 0)$. Let $\mathbf{N}$ be the unit normal of M pointing upward, i.e. $\langle \mathbf{N}, e_3\rangle > 0$ where $e_3 = (0, 0, 1)$. Find the matrix of $\mathcal{S}_p : T_p(M) \to T_p(M)$ with respect to the ordered basis $\{e_1,\ e_2\}$. Find the principal curvatures of M at p.

(5) (Euler formula) Let M be an orientable regular surface with a unit normal vector field $\mathbf{N}$. Let $p \in M$ and let $k_1,\ k_2$ be the principal curvatures with eigenvectors e_1 and e_2 which are orthonormal. Let $\mathbf{v} \in T_p(M)$ such that $\mathbf{v} = \cos\theta e_1 + \sin\theta e_2$. Prove that the normal curvature of the curve α passing through p with tangent vector v is given by

$$k_n = k_1\cos^2\theta + k_2\sin^2\theta.$$

(6) (Gaussian curvature of an ellipsoid) Let M be a regular orientable surface with unit normal vector field $\mathbf{N}$. Let f be a smooth function on M which is nowhere zero. Let $p \in M$ and let $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis for $T_p(M)$.

(i) Prove that the Gaussian curvature of M at p is given by:

$$K = \frac{\langle d(f\mathbf{N})(\mathbf{v}_1) \times d(f\mathbf{N})(\mathbf{v}_2), \mathbf{N}\rangle}{f^2}.$$

Note $d(f\mathbf{N})(\mathbf{v})$ is defined as follow: let α be the curve on M with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$, then

$$d(f\mathbf{N})(\mathbf{v}) = \frac{d}{dt}(f(\alpha(t))\mathbf{N}(\alpha(t)))\big|_{t=0}.$$

(ii) Let M be the ellipsoid

$$h(x, y, z) := \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let f be the restriction of the function

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}}.$$

Apply (i) to show that the Gaussian curvature is given by

$$K = \frac{1}{f^4 a^2 b^2 c^2}.$$

(Hint: We may take $\mathbf{N} = \frac{\nabla h}{|\nabla h|}$. Note that $|\nabla h| = 2f$ and so $d(f\mathbf{N})(\mathbf{v}) = \left(\frac{v_1}{a^2}, \frac{v_2}{b^2}, \frac{v_3}{c^2}\right)$ if $\mathbf{v} = (v_1, v_2, v_3)$.)