Assignment 2, Due 06/10/2020 on or before 11:59 pm

(1) Let $\mathbf{S}^1$ be the unit circle $x^2 + y^2 = 1$. Let $\alpha(s), 0 \leq s \leq 2\pi$, be a parametrization of $\mathbf{S}^1$ by arc length. Let $\mathbf{w}(s) = \alpha'(s) + e_3$ where $e_3 = (0, 0, 1)$. Show the ruled surface

$$\mathbf{X}(s, v) = \alpha(s) + v\mathbf{w}(s)$$

with $-\infty < v < \infty$, is part of the hyperboloid $x^2 + y^2 - z^2 = 1$. Is $\mathbf{X}$ a surjective map to the hyperboloid? Is $\mathbf{X}$ injective? Does $\mathbf{X}$ has rank 2 for $0 < s < 2\pi$, $v \in \mathbb{R}$?

(2) Find a parametrization for the catenoid, which is obtained by revolving the catenary $y = \cosh x$ about the x-axis. Find also the coefficients of first fundamental form.

(3) The Enneper's surface is defined by

$$\mathbf{X}(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$

Show that this a regular surface patch for $u^2 + v^2 < 3$. Also find two points on the circle $u^2 + v^2 = 3$ such that they have the same image under $\mathbf{X}$. Also find a unit normal vector field of the surface.

(4) Prove that the definition of tangent space is independent of the choice of parametrization.

(5) Find the stereographic projection by $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ which maps $\mathbb{R}^2$ onto unit sphere $\mathbb{S}^2$ with the north pole $(0, 0, 1)$ deleted so that for $(u, v) \in \mathbb{R}^2$, $\mathbf{X}(u, v)$ lies on the line segment joining $(0, 0, 1)$ and $(u, v, 0)$. What is the image of the interior of the unit circle under $\mathbf{X}$? What is the image of the exterior of the unit circle? Find also the coefficients of the first fundamental form.

(6) Consider the sphere parametrized by spherical coordinates:

$$\mathbf{X}(u, v) = (\sin v \cos u, \sin v \sin u, \cos v)$$

with $-\pi < u < \pi, 0 < v < \pi$. Find the length of the curve α given by $u = u_0$ and $a \leq v \leq b$ with $0 < a < b < \pi$. (That is $\alpha(t) = (\sin t \cos u_0, \sin t \sin u_0, \cos t)$, with $a \leq t \leq b$.) Let $\beta(t)$ be another curve joining $\alpha(a)$ to $\alpha(b)$ on the surface, i.e. $\beta(t) = \mathbf{X}(u(t), v(t))$, $a \leq t \leq b$ with $\beta(a) = \alpha(a)$, $\beta(b) = \alpha(b)$. Show that $\ell(\beta) \geq \ell(\alpha)$.

(7) Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function defined on an open set U of $\mathbb{R}^3$. Suppose a is a regular value of f. Find a unit normal vector field on $M = \{(x, y, z) | f(x, y, z) = a\}$. Justify

your answer. That is, prove that the vector field you find is perpendicular to the tangent space at every point of M.