Assignment 1, Due 22/9/2020 on or before 11:59 pm

(1) (The tractrix) Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right).$$

(a) Prove that α is regular except at $t = \frac{\pi}{2}$.

(b) Prove that the length of the segment of the tangent of α between the point of tangency and the y-axis is constantly 1.

(2) A regular curve $\alpha(s)$ parametrized by arc length is called a *cylindrical helix* if the is some constant vector $\mathbf{u}$ such that $\langle T, \mathbf{u} \rangle = \cos \theta_0$ is a constant. Prove that a regular curve α parametrized by arc length with $\kappa > 0$ is a cylindrical helix if and only if κ/τ is constant.

(3) Assume that $k(s) > 0$, $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all s for a regular curve $\alpha(s)$ parametrized by arc length. Show that α lies on a sphere if and only if

$$\rho^2 + (\rho')^2 \lambda^2 = \text{constant}.$$

where $\rho = 1/k(s)$, $\lambda = 1/\tau$.

(*Hint*: Necessity: Differentiate $|\alpha|^2$ three times to obtain $\alpha = -\rho N - \rho' \lambda B$. Sufficiency: Show that $\beta = \alpha + \rho N - \rho' \Gamma B$ is constant.)

(4) Let $\alpha : (a, b) \to \mathbb{R}^3$ be a regular smooth curve parametrized by arc length so that the curvature $\kappa(s) > 0$ everywhere. Let $s_0 \in (a, b)$. Show that for $s_3 > s_2 > s_1$ sufficiently close to s_0, $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ do are not collinear. Prove also that as $s_i \to s_0$, the unique plane containing $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ will approaches to the plane spanned by $T(s_0), N(s_0)$.