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If such $\mathbf{N}$ exists, then it is called an orientation of M.

Basic facts

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- $\mathbf{N}$ is smooth means that if $\mathbf{N} = (N_1, N_2, N_3)$ then each N_i is a smooth function.
- $\mathbf{N}$ is continuous and satisfies (ii), (iii) above that $\mathbf{N}$ is smooth.

An instrinsic definition

We have the following intrinsic characterization of orientable surface.

Proposition

M is orientable if and only if there exist coordinate charts covering M so that the change of coordinate matrices have positive determinant.

Proof

Proof: (Sketch)

If M is orientable and $\mathbf{N}$ is an orientation.

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Let $(\mathbf{X}_\alpha, U_\alpha)$ be coordinate charts covering M. If the coordinates of U_α are denoted by (u, v), then we may choose (u, v) so that

$$\mathbf{N} = \frac{(\mathbf{X}_\alpha)_u \times (\mathbf{X}_\alpha)_v}{|(\mathbf{X}_\alpha)_u \times (\mathbf{X}_\alpha)_v|}. \text{ (Why?)}$$

Then these are the coordinate charts we want.

Proof, cont.

Conversely, if $(\mathbf{X}_\alpha, U_\alpha)$ be coordinate charts covering M so that the change of coordinate matrices have positive determinant. Define $\mathbf{N}$ as above, then this gives an orientation of M. (Why?)

A non-orientable surface: the Möbius strip

$$\begin{aligned}\mathbf{X}(\theta, v) =& \underline{(\cos\theta, \sin\theta, 0)} + v \cdot \underline{(\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta)} \\ =& \mathbf{a}(\theta) + v\mathbf{w}(\theta); (-\pi < \theta < \pi, \quad -\frac{1}{2} < v < \frac{1}{2}).\end{aligned}$$

$$\begin{aligned}\lim_{\theta\to-\pi} \mathbf{N}(\theta, 0) &= (0, 0, -1); \\ \lim_{\theta\to\pi} \mathbf{N}(\theta, 0) &= (0, 0, 1).\end{aligned}$$

On the other hand, $\mathbf{x}(\pi, 0) = (-1, 0, 0) = \mathbf{x}(-\pi, 0)$
Hence the Mobiüs strip is not orientable.

Let M be a regular surface in $\mathbb{R}^3$. Suppose M is orientable with orientaion $\mathbf{N}$. That is:

- $\mathbf{N}$ is smooth;
- $\mathbf{N}$ has unit length;
- $\mathbf{N}$ is orthogonal to $T_p(M)$ at all point.

Definition

The shape operator $\mathcal{S}_p$ with respect to $\mathbf{N}$ at p is the operator defined as follows: Let $\mathbf{v} \in T_p(M)$ and let $\alpha(t)$, $-\epsilon < 0 < \epsilon$ be a smooth curve on M with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$. Then $\mathcal{S}_p(\mathbf{v})$ is defined as

$$\mathcal{S}_p(\mathbf{v}) = -\frac{d}{dt}(N(\alpha(t)))\Big|_{t=0}.$$

Remarks

- Notice that there is a negative sign on the RHS in the above.
- $\mathcal{S}_p$ is also called the *Weingarten map* of M at p.
- If $\mathbf{N}$ is a unit normal vector field, then $\mathbf{N}_1 := -\mathbf{N}$ is also a unit normal vector field. The shape operator with respect to $\mathbf{N}_1$ is the negative of the shape operator with respect to $\mathbf{N}$.

Basic facts

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(ii) $\mathcal{S}_p$ *is a linear map from* $T_p(M)$ *to* $T_p(M)$*.*

(iii) $\mathcal{S}_p$ *is self-adjoint with respect to the first fundamental form.*

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Proposition

With the above notation, the following are true:

(i) $\mathcal{S}_p$ *is well-defined.*

(ii) $\mathcal{S}_p$ *is a linear map from* $T_p(M)$ *to* $T_p(M)$.

(iii) $\mathcal{S}_p$ *is self-adjoint with respect to the first fundamental form.*

(vi) $\mathcal{S}$ *is smooth.*

$\mathcal{S}_p$ is well-defined

Proof: (Sketch) Let $\mathbf{X}(u, v)$ be a local parametrization so that $\mathbf{X}(u_0, v_0) = p$. Then $\mathbf{N} = \mathbf{N}(u, v)$.
Let $\alpha(t) = \mathbf{X}(u(t), v(t))$ so that $(u(0), v(0)) = (u_0, v_0)$. Then

$$\frac{d\mathbf{N}(\alpha(t))}{dt} = \mathbf{N}_u u' + \mathbf{N}_v v'.$$

Let $\mathbf{v} = a\mathbf{X}_u + b\mathbf{X}_v$. Now $\mathbf{v} = \alpha'(0) = \mathbf{X}_u u' + \mathbf{X}_v v'$, so $u' = a, v' = b$ at p. Hence

$$\frac{d\mathbf{N}(\alpha(t))}{dt} = a\mathbf{N}_u + b\mathbf{N}_v.$$

So $\mathcal{S}_p$ is well-defined.

$\mathcal{S}_p$ is a linear map from $T_p(M)$ to $T_p(M)$

Note that $\mathbf{N}_u, \mathbf{N}_v$ are in $T_p(M)$ (Why?). So $\mathcal{S}_p : T_p(M) \to T_p(M)$. It is also linear. (Why?)

$\mathcal{S}_p$ is self-adjoint

To prove $\mathcal{S}_p$ is self adjoint. Let $\mathbf{v}, \mathbf{w} \in T_p(M)$. Let $\mathbf{v} = a\mathbf{X}_u + b\mathbf{X}_v$, $\mathbf{w} = c\mathbf{X}_u + d\mathbf{X}_v$. Then

$$\begin{aligned} -\langle \mathcal{S}_p(\mathbf{v}), \mathbf{w} \rangle =& \langle a\mathbf{N}_u + b\mathbf{N}_v, c\mathbf{X}_u + d\mathbf{X}_v \rangle \\ =& ac\langle \mathbf{N}_u, \mathbf{X}_u \rangle + bd\langle \mathbf{N}_v, \mathbf{X}_v + ad\langle \mathbf{N}_u, \mathbf{X}_v \rangle + bc\langle \mathbf{N}_v, \mathbf{X}_u \rangle \end{aligned}$$

$$-\langle \mathcal{S}_p(\mathbf{v}), \mathbf{w} \rangle = ac\langle \mathbf{N}_u, \mathbf{X}_u \rangle + bd\langle \mathbf{N}_v, \mathbf{X}_v + cb\langle \mathbf{N}_u, \mathbf{X}_v \rangle + da\langle \mathbf{N}_v, \mathbf{X}_u \rangle$$

So they are equal. (Why?)

The second fundamental form

Definition

Let $\mathcal{S}$ be the shape operator with respect to a unit normal vector field $\mathbf{N}$, *the second fundamental form* $\mathbb{II}_p$ *of* M *at* p *(with respect to* $\mathbf{N}$*) is the bilinear form* $\mathbb{II}_p(\mathbf{v}, \mathbf{w}) = g(\mathcal{S}_p(\mathbf{v}), \mathbf{w}) = \langle \mathcal{S}_p(\mathbf{v}), \mathbf{w} \rangle$.

Proposition

$\mathbb{II}_p$ *is a symmetric bilinear form on* $T_p(M)$.

With the same notation as in the previous section of M. Let $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$.

Definition

The coefficients of the second fundamental form e, f, g at p are defined as:

$$e = \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_u);$$
$$f = \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_v);$$
$$g = \mathbb{II}_p(\mathbf{X}_v, \mathbf{X}_v).$$

Notation: Suppose we use (u^1, u^2) as coordinates, and $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_2 / |\mathbf{X}_1 \times \mathbf{X}_2|$, then the coefficients of the second fundamental form are denoted by

$$h_{11} = \mathbb{II}_p(\mathbf{X}_1, \mathbf{X}_1);\ h_{12} = \mathbb{II}_p(\mathbf{X}_1, \mathbf{X}_2) = h_{21};\ h_{22} = \mathbb{II}_p(\mathbf{X}_2, \mathbf{X}_2).$$

To compute e, f, g

Proposition

$$\begin{aligned} e =& \langle \mathbf{N}, \mathbf{X}_{uu} \rangle = \frac{\det\left(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu}\right)}{\sqrt{EG - F^2}} \\ f =& \langle \mathbf{N}, \mathbf{X}_{uv} \rangle = \frac{\det\left(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv}\right)}{\sqrt{EG - F^2}}; \\ g =& \langle \mathbf{N}, \mathbf{X}_{vv} \rangle = \frac{\det\left(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv}\right)}{\sqrt{EG - F^2}}. \end{aligned}$$

Suppose $\mathcal{S}_p(\mathbf{X}_u) = a_1^1\mathbf{X}_u + a_1^2\mathbf{X}_v, \mathcal{S}_p(\mathbf{X}_v) = a_2^1\mathbf{X}_u + a_2^2\mathbf{X}_v$. Then the matrix of S_p with respect to the ordered basis $\beta = \{\mathbf{X}_u, \mathbf{X}_v\}$ is given by

$$[\mathcal{S}_p]_\beta = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$$

Definition

The Gaussian curvature $K(p)$ of M at p is the determinant of $\mathcal{S}_p$. The mean curvature $H(p)$ of M at p is $1/2\times$ the trace of $\mathcal{S}_p$.

Proposition

1. *Let*

$$\begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$$

be the matrix of $\mathcal{S}_p$ *with respect to the ordered basis* $\{\mathbf{X}_u, \mathbf{X}_v\}$. *Then*

$$\begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

2. *The Gaussian curvature* $K(p)$ *and the mean curvature* $H(p)$ *of* M *at* p *are given by*

$$K(p) = \frac{eg - f^2}{EG - F^2},$$

and

$$H(p) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

Two remarks

Remark: (i) Gaussian curvature is invariant under reparametrization. (ii) Mean curvature is invariant under *orientation preserving* reparametrization.