The shape operator and the second fundamental form

1. Orientation of regular surfaces

Definition 1. Let M be a regular surface in $\mathbb{R}^3$. M is said to be orientable if there is a unit vector field $\mathbf{N}$ on M such that

(i) $\mathbf{N}$ is smooth;
(ii) $\mathbf{N}$ has unit length;
(iii) $\mathbf{N}$ is orthogonal to $T_p(M)$ at all point.

If such $\mathbf{N}$ exists, then it is called an orientation of M.

Facts:

- If $\mathbf{N}$ is an orientation, then $-\mathbf{N}$ is also an orientation. There are exactly two orientations on an orientable surface.
- $\mathbf{N}$ is smooth means that if $\mathbf{N} = (N_1, N_2, N_3)$ then each N_i is a smooth function.
- $\mathbf{N}$ is continuous and satisfies (ii), (iii) above that $\mathbf{N}$ is smooth.

We have the following intrinsic characterization of orientable surface.

Proposition 1. *M is orientable if and only if there exist coordinate charts covering M so that the change of coordinate matrices have positive determinant.*

Proof. (Sketch) If M is orientable and $\mathbf{N}$ is an orientation. Let $(\mathbf{X}_\alpha, U_\alpha)$ be coordinate charts covering M. If the coordinates of U_α are denoted by (u, v), then we may choose (u, v) so that

$$\mathbf{N} = \frac{(\mathbf{X}_\alpha)_u \times (\mathbf{X}_\alpha)_v}{|(\mathbf{X}_\alpha)_u \times (\mathbf{X}_\alpha)_v|}. \text{ (Why?)}$$

Then these are the coordinate charts we want.

Conversely, if $(\mathbf{X}_\alpha, U_\alpha)$ be coordinate charts covering M so that the change of coordinate matrices have positive determinant. Define $\mathbf{N}$ as above, then this gives an orientation of M. (Why?)

□

2. Möbius strip

$$\mathbf{x}(\theta, v) = (\cos\theta, \sin\theta, 0) + v(\sin\frac{1}{2}\theta\cos\theta, \sin\frac{1}{2}\theta\sin\theta, \cos\frac{1}{2}\theta) = \mathbf{a}(\theta) + v\mathbf{w}(\theta)$$

$-\pi < \theta < \pi, \quad -\frac{1}{2} < v < \frac{1}{2}.$

$$\lim_{\theta\to-\pi} \mathbf{N}(\theta,0) = (0,0,-1);$$
$$\lim_{\theta\to\pi} \mathbf{N}(\theta,0) = (0,0,1).$$

On the other hand, $\mathbf{x}(\pi,0) = (-1,0,0) = \mathbf{x}(-\pi,0)$

Hence the Mobiüs strip is not orientable.

3. The shape operator

Let M be a regular surface in $\mathbb{R}^3$. Suppose M is orientable with orientaion $\mathbf{N}$. That is:

- $\mathbf{N}$ is smooth;
- $\mathbf{N}$ has unit length;
- $\mathbf{N}$ is orthogonal to $T_p(M)$ at all point.

Definition 2. The *shape operator* $\mathcal{S}_p$ with respect to $\mathbf{N}$ at p is the operator defined as follows: Let $\mathbf{v} \in T_p(M)$ and let $\alpha(t)$, $-\epsilon < 0 < \epsilon$ be a smooth curve on M with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$. Then $\mathcal{S}_p(\mathbf{v})$ is defined as

$$\mathcal{S}_p(\mathbf{v}) = -\frac{d}{dt}(N(\alpha(t)))\Big|_{t=0}.$$

Remark:

- Notice that there is a negative sign on the RHS in the above.
- $\mathcal{S}_p$ is also called the Weingarten map of M at p.
- If $\mathbf{N}$ is a unit normal vector field, then $\mathbf{N}_1 := -\mathbf{N}$ is also a unit normal vector field. The shape operator with respect to $\mathbf{N}_1$ is the negative of the shape operator with respect to $\mathbf{N}$.

Proposition 2. *With the above notation, the following are true:*

(i) $\mathcal{S}_p$ *is well-defined.*
(ii) $\mathcal{S}_p$ *is a linear map from* $T_p(M)$ *to* $T_p(M)$.
(iii) $\mathcal{S}_p$ *is self-adjoint with respect to the first fundamental form.*
(vi) $\mathcal{S}$ *is smooth.*

Proof. (Sketch) Let $\mathbf{X}(u,v)$ be a local parametrization so that $\mathbf{X}(u_0,v_0) = p$. Then $\mathbf{N} = \mathbf{N}(u,v)$.

Let $\alpha(t) = \mathbf{X}(u(t),v(t))$ so that $(u(0),v(0)) = (u_0,v_0)$. Then

$$\frac{d\mathbf{N}(\alpha(t))}{dt} = \mathbf{N}_u u' + \mathbf{N}_v v'.$$

Let $\mathbf{v} = a\mathbf{X}_u + b\mathbf{X}_v$. Now $\mathbf{v} = \alpha'(0) = \mathbf{X}_u u' + \mathbf{X}_v v'$, so $u' = a, v' = b$ at p. Hence

$$\frac{d\mathbf{N}(\alpha(t))}{dt} = a\mathbf{N}_u + b\mathbf{N}_v.$$

So $\mathcal{S}_p$ is well-defined.

Note that $\mathbf{N}_u, \mathbf{N}_v$ are in $T_p(M)$ (Why?). So $\mathcal{S}_p : T_p(M) \to T_p(M)$. It is also linear. (Why?)

To prove $\mathcal{S}_p$ is self adjoint. Let $\mathbf{v}, \mathbf{w} \in T_p(M)$. Let $\mathbf{v} = a\mathbf{X}_u + b\mathbf{X}_v$, $\mathbf{w} = c\mathbf{X}_u + d\mathbf{X}_v$. Then

$$\begin{aligned} -\langle \mathcal{S}_p(\mathbf{v}), \mathbf{w}\rangle =& \langle a\mathbf{N}_u + b\mathbf{N}_v, c\mathbf{X}_u + d\mathbf{X}_v\rangle \\ =& ac\langle \mathbf{N}_u, \mathbf{X}_u\rangle + bd\langle \mathbf{N}_v, \mathbf{X}_v + ad\langle \mathbf{N}_u, \mathbf{X}_v\rangle + bc\langle \mathbf{N}_v, \mathbf{X}_u\rangle \end{aligned}$$

$$-\langle \mathcal{S}_p(\mathbf{v}), \mathbf{w}\rangle = ac\langle \mathbf{N}_u, \mathbf{X}_u\rangle + bd\langle \mathbf{N}_v, \mathbf{X}_v + cb\langle \mathbf{N}_u, \mathbf{X}_v\rangle + da\langle \mathbf{N}_v, \mathbf{X}_u\rangle$$

So they are equal. (Why?)

$\mathcal{S}_p$ is smooth. (Why?) □

Definition 3. Let $\mathcal{S}$ be the shape operator with respect to a unit normal vector field $\mathbf{N}$, the *second fundamental form* $\mathbb{II}_p$ of M at p (with respect to $\mathbf{N}$) is the bilinear form $\mathbb{II}_p(\mathbf{v}, \mathbf{w}) = g(\mathcal{S}_p(\mathbf{v}), \mathbf{w}) = \langle \mathcal{S}_p(\mathbf{v}), \mathbf{w}\rangle$.

Proposition 3. *$\mathbb{II}_p$ is a symmetric bilinear form on $T_p(M)$.*

4. Coefficients of the second fundamental form

With the same notation as in the previous section of M. Let $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$.

Definition 4. The *coefficients of the second fundamental form* e, f, g at p are defined as:

$$\begin{aligned} e =& \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_u); \\ f =& \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_v); \\ g =& \mathbb{II}_p(\mathbf{X}_v, \mathbf{X}_v). \end{aligned}$$

Notation: Suppose we use (u^1, u^2) as coordinates, and $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_2 / |\mathbf{X}_1 \times \mathbf{X}_2|$, then the coefficients of the second fundamental form are denoted by

$$h_{11} = \mathbb{II}_p(\mathbf{X}_1, \mathbf{X}_1); h_{12} = \mathbb{II}_p(\mathbf{X}_1, \mathbf{X}_2) = h_{21}; h_{22} = \mathbb{II}_p(\mathbf{X}_2, \mathbf{X}_2).$$

Proposition 4.

$$\begin{aligned} e =& \langle \mathbf{N}, \mathbf{X}_{uu}\rangle = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu})}{\sqrt{EG - F^2}} \\ f =& \langle \mathbf{N}, \mathbf{X}_{uv}\rangle = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uv})}{\sqrt{EG - F^2}}; \\ g =& \langle \mathbf{N}, \mathbf{X}_{vv}\rangle = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv})}{\sqrt{EG - F^2}}. \end{aligned}$$

5. Gaussian curvature and mean curvature

Suppose $\mathcal{S}_p(\mathbf{X}_u) = a_1^1\mathbf{X}_u + a_1^2\mathbf{X}_v, \mathcal{S}_p(\mathbf{X}_v) = a_2^1\mathbf{X}_u + a_2^2\mathbf{X}_v$. Then the matrix of S_p with respect to the ordered basis $\beta = \{\mathbf{X}_u, \mathbf{X}_v\}$ is given by

$$[\mathcal{S}_p]_\beta = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$$

Definition 5. The *Gaussian curvature* $K(p)$ of M at p is the determinant of $\mathcal{S}_p$. The *mean curvature* $H(p)$ of M at p is $1/2\times$the trace of $\mathcal{S}_p$.

Proposition 5. (1) *Let*

$$\begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$$

be the matrix of $\mathcal{S}_p$ with respect to the ordered basis $\{\mathbf{X}_u, \mathbf{X}_v\}$. Then

$$\begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

(2) *The Gaussian curvature $K(p)$ and the mean curvature $H(p)$ of M at p are given by*

$$K(p) = \frac{eg - f^2}{EG - F^2},$$

and

$$H(p) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

Remark: (i) Gaussian curvature is invariant under reparametrization. (ii) Mean curvature is invariant under *orientation preserving* reparametrization.