MATH2230B
Complex Variables with Applications

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Example

To evaluate the Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x} dx,$$

we let $\gamma_{\rho,R}$, $0 < \rho < R$, *be the closed curve consisting of* C_R, $l_{-R,-\rho}$, C_ρ *and* $l_{\rho,R}$ *with counterclockwise orientation, where* C_R *is the upper-half circle centered at origin with radius* R *from* R *to* $-R$, C_ρ *is the upper-half circle centered at origin with radius* ρ *from* $-\rho$ *to* ρ, $l_{-R,-\rho}$ *is the line segment from* $-R$ *to* $-\rho$, *and* $l_{\rho,R}$ *is the line segment from* ρ *to* R. *By Cauchy-Goursat theorem,*

$$\int_{\gamma_{\rho,R}} \frac{e^{iz}}{z} dz = 0.$$

That is,

$$\int_{C_R} \frac{e^{iz}}{z} dz + \int_{l_{-R,-\rho}} \frac{e^{iz}}{z} dz + \int_{C_\rho} \frac{e^{iz}}{z} dz + \int_{l_{\rho,R}} \frac{e^{iz}}{z} dz = 0.$$

Example (continued)

The last equality can be further rewritten as

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x} dx + \int_{\rho}^{R} \frac{e^{ix}}{x} dx = -\int_{C_\rho} \frac{e^{iz}}{z} dz - \int_{C_R} \frac{e^{iz}}{z} dz. \qquad (1)$$

For the left-hand side of (1), we have

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x} dx + \int_{\rho}^{R} \frac{e^{ix}}{x} dx = 2i \int_{\rho}^{R} \frac{\sin x}{x} dx. \qquad (2)$$

For the right-hand side of (1), first, $-C_\rho$ *can be parametrized by* $\rho e^{i\theta}$, $\theta \in [0, \pi]$, *and hence*

$$\int_{C_\rho} \frac{e^{iz}}{z} dz = -\int_0^{\pi} \frac{e^{i\rho e^{i\theta}}}{\rho e^{i\theta}} \cdot i\rho e^{i\theta} d\theta = -i \int_0^{\pi} e^{i\rho e^{i\theta}} d\theta.$$

Example (continued)

By using the fact that

$$|e^z - 1| \leq C|z| \quad \textit{for all } |z| \leq 1$$

for some positive constant C, we have, for $\rho \leq 1$,

$$\left| \int_0^\pi \left(e^{i\rho e^{i\theta}} - 1 \right) d\theta \right| \leq \int_0^\pi \left| e^{i\rho e^{i\theta}} - 1 \right| d\theta \leq \int_0^\pi C\rho d\theta = C\pi\rho.$$

Thus,

$$\lim_{\rho \to 0} \int_0^\pi e^{i\rho e^{i\theta}} d\theta = \pi,$$

which gives

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{e^{iz}}{z} dz = -\pi i.$$

Example (continued)

Second, by Jordan's lemma,

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z} dz = 0.$$

By (2) and the last two equalities, (1) implies

$$2i \int_0^\infty \frac{\sin x}{x} dx = \pi i.$$

That is,

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Example

For $-1 < a < 3$, to evaluate the integral

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}dx,$$

we let

$$f(z) = \frac{z^a}{(z^2+1)^2},$$

where $z^a = e^{a \log z}$ is defined by using the branch of logarithm

$$\log z = \ln|z| + i\theta, \quad \theta \in \arg z, \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

In addition, we let $\gamma_{\rho,R}$, C_R, C_ρ, $l_{-R,-\rho}$ and $l_{\rho,R}$ be defined as in the last example with $0 < \rho < 1 < R$. Since i is the only singularity of f in the region enclosed by $\gamma_{\rho,R}$, by residue theorem,

$$\int_{\gamma_{\rho,R}} f(z)dz = 2\pi i \operatorname{Res}(f(z); i). \tag{3}$$

Example (continued)

By letting

$$f(z) = \frac{g(z)}{(z-i)^2}, \quad \text{where } g(z) = \frac{z^a}{(z+i)^2},$$

we have

$$\mathrm{Res}\,(f(z); i) = g'(i) = \left.\frac{(a-2)z^a + iaz^{a-1}}{(z+i)^3}\right|_{z=i} = \frac{1-a}{4}\, e^{(a-1)\log i}$$
$$= ie^{ia\pi/2}\left(\frac{a-1}{4}\right).$$

We can divide the integral along $\gamma_{\rho,R}$ into four parts.

$$\int_{\gamma_{\rho,R}} f(z)dz = \int_{C_R} \frac{z^a}{(z^2+1)^2}dz + \int_{l_{-R,-\rho}} \frac{z^a}{(z^2+1)^2}dz$$
$$+ \int_{C_\rho} \frac{z^a}{(z^2+1)^2}dz + \int_{l_{\rho,R}} \frac{z^a}{(z^2+1)^2}dz.$$

Example (continued)

First,

$$\int_{l_{\rho,R}} \frac{z^a}{(z^2+1)^2}dz = \int_{\rho}^{R} \frac{x^a}{(x^2+1)^2}dx.$$

Second,

$$\int_{l_{-R,-\rho}} \frac{z^a}{(z^2+1)^2}dz = \int_{-R}^{-\rho} \frac{e^{a(\ln|x|+i\pi)}}{(x^2+1)^2}dx = e^{ia\pi}\int_{\rho}^{R} \frac{x^a}{(x^2+1)^2}dx.$$

Third,

$$\begin{aligned}\int_{C_\rho} \frac{z^a}{(z^2+1)^2}dz &= -\int_{-C_\rho} \frac{z^a}{(z^2+1)^2}dz \\ &= -\int_0^{\pi} \frac{e^{a\log\left(\rho e^{i\theta}\right)}}{\left(\rho^2 e^{2i\theta}+1\right)^2}\cdot i\rho e^{i\theta}d\theta \\ &= -i\rho\int_0^{\pi} \frac{e^{a(\ln\rho+i\theta)}}{\left(\rho^2 e^{2i\theta}+1\right)^2}\cdot e^{i\theta}d\theta \\ &= -i\rho^{a+1}\int_0^{\pi} \frac{e^{i(a+1)\theta}}{\left(\rho^2 e^{2i\theta}+1\right)^2}d\theta.\end{aligned}$$

Example (continued)

Thus,

$$\left| \int_{C_\rho} \frac{z^a}{(z^2+1)^2} dz \right| \leq \frac{\pi \rho^{a+1}}{(1-\rho^2)^2},$$

which implies

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{z^a}{(z^2+1)^2} dz = 0.$$

Finally,

$$\begin{aligned} \int_{C_R} \frac{z^a}{(z^2+1)^2} dz &= \int_0^\pi \frac{e^{a \log\left(Re^{i\theta}\right)}}{\left(R^2 e^{2i\theta}+1\right)^2} \cdot iRe^{i\theta} d\theta \\ &= iR \int_0^\pi \frac{e^{a(\ln R + i\theta)}}{\left(R^2 e^{2i\theta}+1\right)^2} \cdot e^{i\theta} d\theta \\ &= iR^{a+1} \int_0^\pi \frac{e^{i(a+1)\theta}}{\left(R^2 e^{2i\theta}+1\right)^2} d\theta. \end{aligned}$$

Example (continued)

Thus,

$$\left| \int_{C_R} \frac{z^a}{(z^2+1)^2} dz \right| \le \frac{\pi R^{a+1}}{\left(R^2-1\right)^2},$$

which implies

$$\lim_{R\to\infty} \int_{C_R} \frac{z^a}{(z^2+1)^2} dz = 0.$$

By passing to the limit $\rho \to 0$ and $R \to \infty$, (3) becomes

$$\begin{aligned}\left(1+e^{ia\pi}\right) \int_0^{\infty} \frac{x^a}{(x^2+1)^2} dx &= 2\pi i \cdot i e^{ia\pi/2} \left(\frac{a-1}{4}\right) \\ &= \frac{\pi(1-a)e^{ia\pi/2}}{2}.\end{aligned}$$

Example (continued)

Therefore, if $a \neq 1$,

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)e^{ia\pi/2}}{2\left(1+e^{ia\pi}\right)} = \frac{\pi(1-a)}{2\left(e^{-ia\pi/2}+e^{ia\pi/2}\right)}$$
$$= \frac{\pi(1-a)}{4\cos(a\pi/2)}.$$

For $a = 1$, by change of variables $y = x^2+1$,

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \int_0^\infty \frac{x}{(x^2+1)^2}\,dx$$
$$= \lim_{R\to\infty}\int_0^R \frac{x}{(x^2+1)^2}\,dx$$
$$= \frac{1}{2}\lim_{R\to\infty}\int_1^{R^2+1}\frac{dy}{y^2}$$
$$= \frac{1}{2}\lim_{R\to\infty}\left(1-\frac{1}{R^2+1}\right) = \frac{1}{2}.$$