MATH2230B
Complex Variables with Applications

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Remark (L'Hospital rule)

If f and g are analytic at z_0 and $f(z_0) = g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$

provided the limit on the right-hand side exists.

Example

Let $f(z) = \dfrac{z - \sinh z}{z^2 \sinh z}$*. Singularities of* f *occur at* $z = n\pi i$, $n \in \mathbb{Z}\backslash\{0\}$*. At each* $z = n\pi i$, $n \in \mathbb{Z}\backslash\{0\}$*, since*

$$n\pi i - \sinh n\pi i = n\pi i \neq 0$$

and

$$\begin{aligned} \left(z^2 \sinh z\right)' \Big|_{z=n\pi i} &= 2n\pi i \sinh n\pi i + (n\pi i)^2 \cosh n\pi i \\ &= (-1)^{n+1} n^2 \pi^2 \neq 0, \end{aligned}$$

$z = n\pi i$ *is a simple pole of* f*. Let*

$$f(z) = \frac{g(z)}{z - n\pi i}, \quad \text{where } g(z) = \frac{(z - n\pi i)(z - \sinh z)}{z^2 \sinh z}.$$

Then

$$\mathrm{Res}(f; n\pi i) = \lim_{z \to n\pi i} g(z) = \frac{(-1)^{n+1} i}{n\pi}.$$

Definition

For a continuous real-valued function f defined on $[0, \infty)$ or $\mathbb{R}$, the improper integral of f is defined by

$$\int_0^{\infty} f(x)dx = \lim_{R \to \infty} \int_0^R f(x)dx$$

and

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2 \to \infty} \int_0^{R_2} f(x)dx, \qquad (1)$$

respectively, provided the limits on the right-hand sides of the equalities exist. There is another value assigned to the improper integral in (1), called the Cauchy principal value of the integral, and defined by

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx.$$

Example

To evaluate the integral

$$\int_0^\infty \frac{dx}{x^6+1},$$

firstly, we let γ_R, $R > 1$, *be the closed curve consisting of* C_R, *the upper-half circle centered at the origin with radius* R, *and* l_R, *the line segment from* $-R$ *to* R. *And we assume that* γ_R *is counterclockwise oriented. In the region enclosed by* γ_R, *there are three zeros of* x^6+1, *that is,* $c_1 = e^{i\pi/6}$, $c_2 = e^{i3\pi/6} = i$ *and* $c_3 = e^{i5\pi/6}$. *By residue theorem,*

$$\int_{\gamma_R} \frac{dz}{z^6+1} = 2\pi i \sum_{k=1}^{3} \mathrm{Res}\left(\frac{1}{z^6+1}; c_k\right).$$

For each $k = 1, 2, 3$, c_k *is a simple pole of* $\dfrac{1}{z^6+1}$, *and we have*

$$\mathrm{Res}\left(\frac{1}{z^6+1}; c_k\right) = \lim_{z\to c_k} \frac{z-c_k}{z^6+1} = \frac{1}{6c_k^5} = -\frac{c_k}{6}.$$

Example (continued)

Therefore,

$$\int_{\gamma_R} \frac{dz}{z^6+1} = \frac{2\pi}{3}.$$

Notice that

$$\int_{l_R} \frac{dz}{z^6+1} = \int_{-R}^{R} \frac{dx}{x^6+1}.$$

And we have

$$\left| \int_{C_R} \frac{dz}{z^6+1} \right| \leq \frac{\pi R}{R^6-1} \longrightarrow 0 \qquad \text{as } R \to \infty.$$

By passing to the limit $R \to \infty$*,*

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.$$

Since $\dfrac{1}{x^6+1}$ *is even, we have*

$$\int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}.$$

Example

Now, we want to evaluate integrals of the form

$$\int_{-\infty}^{\infty} f(x) \sin ax \, dx \quad or \quad \int_{-\infty}^{\infty} f(x) \cos ax \, dx, \quad a > 0.$$

In view of Euler's formula, it is equivalent to consider

$$\int_{-\infty}^{\infty} f(x) e^{iax} dx.$$

These integrals occur in the theory of Fourier analysis. Let γ_R, C_R and l_R be defined as in the last example. If $z_1, ..., z_N$ are all the singularities of $f(z)e^{iaz}$ in the region enclosed by γ_R for R large. By residue theorem,

$$\int_{\gamma_R} f(z) e^{iaz} dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}\left(f(z)e^{iaz}; z_k\right).$$

Therefore, we have

$$\begin{aligned} \int_{-R}^{R} f(x) e^{iax} dx &= \int_{l_R} f(z) e^{iaz} dz \\ &= 2\pi i \sum_{k=1}^{N} \mathrm{Res}\left(f(z)e^{iaz}; z_k\right) - \int_{C_R} f(z) e^{iaz} dz. \end{aligned} \tag{2}$$

Example

To evaluate the integral

$$\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx,$$

we follow the last example with

$$f(z) = \frac{1}{(z^2+4)^2} \quad \textit{and} \quad a = 2.$$

Notice that $2i$ is the only singularity of $\dfrac{e^{i2z}}{(z^2+4)^2}$ in the region enclosed by γ_R for R large. Then (2) becomes

$$\int_{-R}^{R} \frac{e^{i2x}}{(x^2+4)^2}\,dx = 2\pi i \operatorname{Res}\left(\frac{e^{i2z}}{(z^2+4)^2}; 2i\right) - \int_{C_R} \frac{e^{i2z}}{(z^2+4)^2}\,dz.$$

Example (continued)

On one hand, $2i$ *is a pole of order 2 of* $\frac{e^{i2z}}{(z^2+4)^2}$. *By letting*

$$\frac{e^{i2z}}{(z^2+4)^2} = \frac{g(z)}{(z-2i)^2}, \quad \text{where } g(z) = \frac{e^{i2z}}{(z+2i)^2},$$

we have

$$\mathrm{Res}\left(\frac{e^{i2z}}{(z^2+4)^2}; 2i\right) = g'(2i) = \frac{5}{32e^4 i}.$$

On the other hand,

$$\left|\int_{C_R} \frac{e^{i2z}}{(z^2+4)^2} dz\right| \leq \frac{\pi R}{(R^2-4)^2} \longrightarrow 0 \quad \text{as } R \to \infty.$$

Therefore, by passing to the limit $R \to \infty$,

Example (continued)

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i2x}}{(x^2+4)^2} dx = 2\pi i \cdot \frac{5}{32e^4 i} = \frac{5\pi}{16e^4}.$$

Taking the real parts on both sides above yields

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+4)^2} dx = \frac{5\pi}{16e^4}.$$

Since $\frac{\cos 2x}{(x^2+4)^2}$ *is even,*

$$\int_{0}^{\infty} \frac{\cos 2x}{(x^2+4)^2} dx = \frac{5\pi}{32e^4}.$$

Lemma (Jordan's lemma)

Let C_R be defined as in the last example. Suppose that

(i) *f is analytic on $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0, |z| \geq R_0\}$ for some $R_0 > 0$;*

(ii) *For each $R > R_0$, there is a positive constant M_R such that*

$$\max_{C_R} |f| \leq M_R,$$

and

$$\lim_{R \to \infty} M_R = 0.$$

Then, for every $a > 0$,

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{iaz} dz = 0.$$

Proof.

For $a > 0$, $R > R_0$,

$$\int_{C_R} f(z)e^{iaz}dz = \int_0^{\pi} f\left(Re^{i\theta}\right) e^{iaRe^{i\theta}} \cdot iRe^{i\theta} d\theta$$
$$= iR \int_0^{\pi} f\left(Re^{i\theta}\right) e^{-aR\sin\theta} e^{iaR\cos\theta} e^{i\theta} d\theta.$$

Thus,

$$\left| \int_{C_R} f(z)e^{iaz}dz \right| \leq RM_R \int_0^{\pi} e^{-aR\sin\theta} d\theta.$$

Notice that

$$\int_0^{\pi} e^{-aR\sin\theta} d\theta = 2 \int_0^{\pi/2} e^{-aR\sin\theta} d\theta.$$

Proof, continued.

By using the fact that $\sin\theta \geq \frac{2\theta}{\pi}$ for $\theta \in \left[0, \frac{\pi}{2}\right]$,

$$\int_0^{\pi/2} e^{-aR\sin\theta} d\theta \leq \int_0^{\pi/2} e^{-2aR\theta/\pi} d\theta = \frac{\pi}{2aR}\left(1 - e^{-aR}\right) \leq \frac{\pi}{2aR}.$$

Therefore,

$$\left|\int_{C_R} f(z)e^{iaz} dz\right| \leq RM_R \cdot \frac{\pi}{aR} \longrightarrow 0 \quad \text{as } R \to \infty.$$

□

Example

To evaluate the integral

$$\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx,$$

we follow the argument for the integral $\displaystyle\int_{-\infty}^{\infty} f(x)\sin ax\,dx$ *with*

$$f(z)=\frac{z}{z^2+3} \quad \textit{and} \quad a=2.$$

Notice that $\sqrt{3}i$ *is the only singularity of* $\dfrac{ze^{i2z}}{z^2+3}$ *enclosed by* γ_R *for* R *large. Then we have*

$$\int_{-R}^{R}\frac{xe^{i2x}}{x^2+3}\,dx = 2\pi i\,\mathrm{Res}\left(\frac{ze^{i2z}}{z^2+3};\sqrt{3}i\right)-\int_{C_R}\frac{ze^{i2z}}{z^2+3}\,dz.$$

Example (continued)

Let

$$\frac{ze^{i2z}}{z^2+3} = \frac{g(z)}{z-\sqrt{3}i}, \quad \textit{where } g(z) = \frac{ze^{i2z}}{z+\sqrt{3}i},$$

then

$$\mathrm{Res}\left(\frac{ze^{i2z}}{z^2+3}; \sqrt{3}i\right) = g\left(\sqrt{3}i\right) = \frac{1}{2e^{2\sqrt{3}}}.$$

Moreover,

$$\max_{C_R} |f| \leq \frac{R}{R^2-3} \longrightarrow 0 \quad \textit{as } R \to \infty.$$

By Jordan's lemma,

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{i2z}}{z^2+3} dz = 0.$$

Example (continued)

Therefore,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{xe^{i2x}}{x^2+3}dx = 2\pi i \cdot \frac{1}{2e^{2\sqrt{3}}} = \frac{\pi i}{e^{2\sqrt{3}}}.$$

Taking the imaginary parts on both sides above yields

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+3}dx = \frac{\pi}{e^{2\sqrt{3}}}.$$

Since $\dfrac{x\sin 2x}{x^2+3}$ *is even,*

$$\int_{0}^{\infty} \frac{x\sin 2x}{x^2+3}dx = \frac{\pi}{2e^{2\sqrt{3}}}.$$