MATH2230B
Complex Variables with Applications

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Definition

If f is analytic and has the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

on a punctured neighborhood $B_\sigma(z_0) \backslash \{z_0\}$, $\sigma > 0$. Then the coefficient c_{-1} is called the residue of f at z_0, denoted by

$$\mathrm{Res}(f; z_0) = c_{-1}.$$

Proposition

Let γ be the circle centered at z_0 with radius R and counterclockwise orientation. If f is analytic on $\overline{B_R(z_0)} \backslash \{z_0\}$, $z_0 \in \Omega$, then

$$\int_\gamma f(z)dz = 2\pi i \,\mathrm{Res}(f; z_0).$$

Proof.

Recall that the coefficient of Laurent series

$$c_{-1} = \frac{1}{2\pi i} \int_\gamma f(z)dz.$$

Equivalently,

$$\int_\gamma f(z)dz = 2\pi i \, c_{-1} = 2\pi i \,\mathrm{Res}(f; z_0).$$

□

Corollary (residue theorem)

Let Ω be the open set enclosed by a simple closed curve γ with counterclockwise orientation. If f is analytic on $\overline{\Omega}\backslash\{z_1, ..., z_N\}$ for N distinct points $z_1, ..., z_N \in \Omega$, then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f; z_k).$$

Proof.

By choosing $\delta > 0$ sufficiently small, we have $\overline{B_\delta(z_k)}$, $k = 1, ..., N$, are contained in Ω and mutually disjoint. Let γ_k be the boundary of $B_\delta(z_k)$ with counterclockwise orientation. Then

$$\int_\gamma f(z)dz = \sum_{k=1}^{N} \int_{\gamma_k} f(z)dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f; z_k).$$

□

Remark

(i) *If f is analytic at z_0, then* $\mathrm{Res}(f; z_0) = 0$.

(ii) *Suppose that f has a pole of order less than or equal to m at z_0 for some $m \in \mathbb{N}$. The function*

$$g(z) = (z - z_0)^m f(z)$$

has a removable singularity at z_0. We can extend the domain of g to include z_0 and have the Taylor series about z_0,

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{g^{(n)}(z_0)}{n!},$$

in a neighborhood of z_0. And hence, the Laurent series of f about z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-m} = \sum_{k=-m}^{\infty} a_{k+m} (z - z_0)^k$$

near z_0. The residue of f at z_0 is

$$\mathrm{Res}(f; z_0) = a_{m-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

Example

Let $f(z) = \frac{e^z - 1}{z^4}$. *We are going to compute* $\mathrm{Res}(f; 0)$.

$$f(z) = \frac{g(z)}{z^4}, \quad \textit{where } g(z) = e^z - 1.$$

Thus,

$$\mathrm{Res}(f; 0) = \frac{g^{(3)}(0)}{3!} = \frac{1}{6}.$$

And by the residue theorem

$$\int_\gamma f(z)dz = 2\pi i \,\mathrm{Res}(f; 0) = \frac{\pi i}{3}$$

for any simple closed curve γ *surrounding* 0.

Example

Let $f(z) = \dfrac{1}{z(z-2)^5}$. *In order to evaluate the integral*

$$\int_\gamma f(z)dz,$$

where γ *is the unit circle centered at 2 with counterclockwise orientation, we first compute* $\mathrm{Res}(f;2)$. *Since*

$$f(z) = \frac{g(z)}{(z-2)^5}, \quad \text{where } g(z) = \frac{1}{z}.$$

Thus,

$$\mathrm{Res}(f;2) = \frac{g^{(4)}(2)}{4!} = \frac{1}{32}.$$

By the residue theorem

$$\int_\gamma f(z)dz = 2\pi i \, \mathrm{Res}(f;2) = \frac{\pi i}{16}.$$

Example

Let $f(z) = \dfrac{1}{z(z-2)^5}$*. In order to evaluate the integral*

$$\int_\gamma f(z)dz,$$

where γ *is the circle centered at* 2 *with radius* 5*, counterclockwise oriented, we need to compute* $\mathrm{Res}(f;2)$ *and* $\mathrm{Res}(f;0)$*. In the last example, we showed*

$$\mathrm{Res}(f;2) = \frac{1}{32}.$$

To compute $\mathrm{Res}(f;0)$*, we have*

$$f(z) = \frac{h(z)}{z}, \quad \textit{where } h(z) = \frac{1}{(z-2)^5}.$$

Thus,

$$\mathrm{Res}(f;0) = \frac{h(0)}{0!} = -\frac{1}{32}.$$

By the residue theorem

$$\int_\gamma f(z)dz = 2\pi i\left(\mathrm{Res}(f;0) + \mathrm{Res}(f;2)\right) = 0.$$

Example

Let $f(z) = \dfrac{1}{1-\cos z}$. *Notice that*

$$1 - \cos z = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} z^{2k}.$$

Hence, f *has a pole of order* 2 *at* 0*. Let*

$$f(z) = \frac{g(z)}{z^2}, \quad \textit{where } g(z) = \frac{z^2}{1-\cos z}.$$

By L'Hospital rule,

$$\lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{2z}{\sin z} = \lim_{z \to 0} \frac{2}{\cos z} = 2.$$

The residue of f *at* 0 *can be computed by*

$$\mathrm{Res}(f;0) = \lim_{z \to 0} \frac{\dfrac{z^2}{1-\cos z} - 2}{z - 0} = 0.$$

Remark (L'Hospital rule)

If f and g are analytic at z_0 and $f(z_0) = g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$

provided the limit on the right-hand side exists.

Example

Let $f(z) = \cot z = \dfrac{\cos z}{\sin z}$. *Singularities of* f *occur at* $z = n\pi$, $n \in \mathbb{Z}$. *For each* $z = n\pi$, $n \in \mathbb{Z}$, *it is a simple pole, i.e., pole of order* 1, *of* $\dfrac{1}{\sin z}$, *and hence a simple pole of* f. *Let*

$$f(z) = \frac{g(z)}{z - n\pi}, \quad \text{where } g(z) = \frac{(z - n\pi)\cos z}{\sin z}.$$

Then

$$\mathrm{Res}(f; n\pi) = \lim_{z \to n\pi} \frac{(z - n\pi)\cos z}{\sin z} = 1.$$