MATH2230B
Complex Variables with Applications

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In the following, we assume that f is analytic on a punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$. f can be represented as Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad 0 < |z - z_0| < R. \tag{1}$$

where $c_n \in \mathbb{C}$, $n \in \mathbb{Z}$. There are three types of singularities: removable singularities, poles, and essential singularities.

Definition

Let f be analytic on a punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$ with representation (1).

(i) *If $c_n = 0$ for all $n < 0$, then z_0 is called a removable singularity.*

(ii) *If there is $m \in \mathbb{N}$ such that $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$, then z_0 is called a pole of order m.*

(iii) *If there are infinitely many $c_n \neq 0$ with $n < 0$, then z_0 is called an essential singularity.*

Example

(i) *Let $f(z) = \dfrac{1}{z(1+z^2)}$ be defined on $B_1(0)\backslash\{0\}$, then 0 is a pole of f.*

(ii) *Let $f(z) = e^{1/z}$ be defined on $B_1(0)\backslash\{0\}$, then 0 is an essential singularity of f.*

Proposition

If z_0 is a removable singularity of f, by defining

$$g(z) = \begin{cases} f(z), & 0 < |z - z_0| < R, \\ c_0, & z = z_0, \end{cases}$$

then the function g is analytic on $B_R(z_0)$.

Proof.

Notice that

$$g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has the radius of convergence not less than R, otherwise f cannot be defined on $B_R(z_0) \backslash \{z_0\}$. Since g is a power series, g is analytic on its disc of convergence, which contains $B_R(z_0)$. □

Proposition

If f is bounded and analytic on $B_R(z_0)\backslash\{z_0\}$. Then z_0 is a removable singularity of f.

Proof.

Notice that f can be represented as its Laurent series with the coefficients

$$c_n = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(w)}{(w-z_0)^{n+1}}dw, \quad n \in \mathbb{Z},$$

where γ_ρ is the circle centered at z_0 with radius ρ and counterclockwise orientation, for any $\rho > 0$. For each $n < 0$,

$$|c_n| \leq \frac{1}{2\pi} \cdot \frac{\sup|f|}{\rho^{n+1}} \cdot 2\pi\rho = \frac{\sup|f|}{\rho^n}.$$

Since ρ is arbitrary, we conclude that $c_n = 0$ for all $n < 0$. That is, z_0 is a removable singularity. □

Proposition

If z_0 is a pole of f, then

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Proof.

If z_0 is a pole of order m, $m \in \mathbb{N}$, we have

$$f(z) = \sum_{n=-m}^{\infty} c_n (z - z_0)^n, \quad 0 < |z - z_0| < R,$$

where $c_{-m} \neq 0$. By letting

$$g(z) = (z - z_0)^m f(z) = \sum_{n=0}^{\infty} c_{n-m} (z - z_0)^n,$$

which is analytic on $B_R(z_0) \backslash \{z_0\}$, then z_0 is a removable singularity of g with

$$\lim_{z \to z_0} g(z) = c_{-m} \neq 0.$$

Therefore,

$$|f(z)| = \frac{|g(z)|}{|z - z_0|^m} \longrightarrow \infty \quad \text{as } z \to z_0.$$

□

Proposition

If z_0 *is an essential singularity of* f*, then for any* $c \in \mathbb{C}$*, there is a sequence* $z_k \to z_0$ *such that*

$$|f(z_k) - c| \longrightarrow 0 \quad \text{as } k \to \infty.$$

Proof.

Suppose on the contrary that there are $c \in \mathbb{C}$, $\varepsilon_0 > 0$ and $\delta_0 \in (0, R)$ such that

$$|f(z) - c| > \varepsilon_0 \quad \text{for all } z \in B_{\delta_0}(z_0) \backslash \{z_0\}.$$

We define

$$g(z) = \frac{1}{f(z) - c} \quad \text{on } B_{\delta_0}(z_0) \backslash \{z_0\}.$$

Then g is analytic and

$$|g(z)| \le \frac{1}{|f(z) - c|} \le \frac{1}{\varepsilon_0} \quad \text{on } B_{\delta_0}(z_0) \backslash \{z_0\}.$$

Therefore, z_0 is a removable singularity of g. That is,

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{on } B_{\delta_0}(z_0) \backslash \{z_0\}$$

for some $a_n \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$.

Proof, continued.

If $a_0 \neq 0$, we have

$$\lim_{z \to z_0} g(z) = a_0 \neq 0,$$

which implies that

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{1}{g(z)} + c = \frac{1}{a_0} + c.$$

Therefore, f is bounded on $B_\sigma(z_0) \backslash \{z_0\}$ for some $\sigma > 0$. Thus, again, z_0 is a removable singularity of f, a contradiction. We must have $a_0 = 0$. Let N be the least number (if it exists) such that

$$a_0 = a_1 = ... = a_{N-1} = 0 \quad \text{and} \quad a_N \neq 0.$$

Define

$$h(z) = \sum_{n=0}^{\infty} a_{n+N}(z - z_0)^n \quad \text{on } B_{\delta_0}(z_0),$$

which is analytic with $h(z_0) = a_N \neq 0$.

Proof, continued.

Then

$$g(z) = (z - z_0)^N \sum_{n=N}^{\infty} a_n (z - z_0)^{n-N} = (z - z_0)^N h(z)$$

on $B_{\delta_0}(z_0) \backslash \{z_0\}$. Moreover, on $B_{\delta_0}(z_0) \backslash \{z_0\}$,

$$f(z) = \frac{1}{g(z)} + c = \frac{1}{h(z)(z - z_0)^N} + c,$$

which implies that z_0 is a pole of order N of f, again a contradiction. As a consequence, $a_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$, which implies that

$$g(z) = 0 \quad \text{on } B_{\delta_0}(z_0) \backslash \{z_0\}.$$

It is still impossible. We then complete the proof. □

Theorem (Picard's theorem)

If z_0 is an essential singularity of f, then on any punctured neighborhood of z_0, f takes all complex values, with at most one exception, infinitely often.

Example

Let $f(z) = e^{1/z}$. *Then* 0 *is an essential singularity of* f. *The value* 0 *is the only exceptional value which cannot be taken by* f *on any punctured neighborhood of the point* 0. *For any non-zero complex value* $c = \rho e^{i\theta}$, *we solve the equation*

$$e^{1/z} = e^{\frac{1}{|z|^2}(x - iy)} = \rho e^{i\theta} = c, \quad z = x + yi. \tag{2}$$

We have

$$e^{x/|z|^2} = \rho \quad \textit{and} \quad e^{-iy/|z|^2} = e^{i\theta}.$$

That is,

$$\frac{x}{|z|^2} = \ln \rho \quad \textit{and} \quad \frac{y}{|z|^2} = -\theta + 2n\pi, \quad n \in \mathbb{Z}.$$

Example (continued)

The last two equations imply

$$\frac{1}{|z|^2} = (\ln \rho)^2 + (-\theta + 2n\pi)^2 . \tag{3}$$

Thus, we know that for each $n \in \mathbb{Z}$, $z_n = x_n + y_n i$ *is a solution of (2), where*

$$x_n = \frac{\ln \rho}{(\ln \rho)^2 + (-\theta + 2n\pi)^2} \quad \textit{and} \quad y_n = \frac{-\theta + 2n\pi}{(\ln \rho)^2 + (-\theta + 2n\pi)^2}.$$

By (3), $z_n \to 0$ *as* $n \to \infty$*. That is,* c *can be taken by* f *infinitely many times on any punctured neighborhood of* 0*.*

Definition

Let f be defined on an open connected set Ω. The image of a set $X \subset \Omega$ under f is defined by

$$f(X) = \{w \in \mathbb{C} : w = f(z) \text{ for some } z \in X\}.$$

The preimage of a set $Y \subset f(\Omega)$ under f is defined by

$$f^{-1}(Y) = \{z \in \mathbb{C} : f(z) = w \text{ for some } w \in Y\}.$$

If $Y = \{c\}$, we may write $f^{-1}(Y) = f^{-1}(c)$ for simplicity.

Proposition

If f is non-constant, analytic, and $f(z_0) = c$ for some $z_0 \in \Omega$, then there is $\varepsilon > 0$ such that

$$B_\varepsilon(z_0) \cap f^{-1}(c) = \{z_0\}.$$

Proof.

Near z_0, f has the Taylor series

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} a_n (z - z_0)^n = c + \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Since f is non-constant, there is a smallest $N \in \mathbb{N}$ such that $a_N \neq 0$. And we can rewrite above representation as

$$f(z) = c + \sum_{n=N}^{\infty} a_n (z - z_0)^n = c + (z - z_0)^N g(z),$$

where

$$g(z) = \sum_{n=0}^{\infty} a_{N+n} (z - z_0)^n.$$

Since $a_N \neq 0$, there is $\varepsilon > 0$ such that

$$g(z) \neq 0 \quad \text{on } B_\varepsilon(z_0),$$

which implies that

$$f(z) \neq c \quad \text{on } B_\varepsilon(z_0) \backslash \{z_0\}.$$

□

Theorem

Suppose that f is analytic on an open connected set Ω. If $f(z_n) = 0$, where $z_n \in \Omega$ is a sequence of distinct points with a limit point in Ω, then f is identical to 0.

Proof.

By taking a subsequence, still indexed by n, $z_0 = \lim_{n\to\infty} z_n$. By the continuity of f, $f(z_0) = 0$. Suppose that f is not identical to 0, by the isolation of points in preimage, there is $\varepsilon > 0$ such that

$$B_\varepsilon(z_0) \cap f^{-1}(0) = \{z_0\},$$

a contradiction. □

Corollary

Suppose that f and g are analytic on an open connected set Ω. If $f(z_n) = g(z_n)$, where $z_n \in \Omega$ is a sequence of distinct points with a limit point in Ω, then f is identical to g.

Remark

If f and F are analytic on Ω' and Ω, respectively, where $\Omega' \subset \Omega$. If $f(z) = F(z)$ on Ω', then F is an analytic continuation of f. This corollary guarantees that there can be only one such analytic continuation. In particular, suppose that f_1 and f_2 are analytic on Ω_1 and Ω_2, respectively, and $f_1 = f_2$ on $\Omega_1 \cap \Omega_2 \neq \phi$. Then the function

$$g(z) = \begin{cases} f_1(z) & \text{if } z \in \Omega_1 \backslash \Omega_2, \\ f_2(z) & \text{if } z \in \Omega_2, \end{cases}$$

on $\Omega_1 \cup \Omega_2$ is an analytic continuation of both f_1 and f_2.

Theorem (reflection principle)

Let Ω be an open connected set which is symmetric with respect to the real axis. $\Omega = \Omega^+ \cup I \cup \Omega^-$, where Ω^+ is the upper half part, Ω^- is the lower half part, and $I = \Omega \cap \mathbb{R}$. If f is analytic on Ω, then

$$\overline{f(z)} = f(\bar{z}), \quad z \in \Omega, \tag{4}$$

if and only if f is real-valued on I.

Proof.

Suppose that (4) holds. For $x \in I$, we have

$$\overline{f(x)} = f(\bar{x}) = f(x),$$

which gives $f(x) \in \mathbb{R}$. On the other hand, if f is real-valued on I, we can define

$$g(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \cup I, \\ \overline{f(\bar{z})} & \text{if } z \in \Omega^-. \end{cases}$$

Proof, continued.

Then g is analytic on Ω^+. For each $z_0 \in \Omega^-$, we have $\overline{z_0} \in \Omega^+$, and hence

$$g(z) = \sum_{n=0}^{\infty} a_n (z - \overline{z_0})^n$$

in a neighborhood of $\overline{z_0}$ in Ω^+. By the definition of g,

$$g(z) = \sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n$$

in a neighborhood of z_0 in Ω^-. That is, g is analytic on Ω^-. And for each $x_0 \in I$, we have

$$f(z) = \sum_{n=0}^{\infty} b_n (z - x_0)^n$$

in a neighborhood of x_0, say $B_\delta(x_0)$. In addition, b_n's are all real since f takes real values on I.

Proof, continued.

Hence,

$$g(z) = \sum_{n=0}^{\infty} b_n (z - x_0)^n \quad \text{on } B_\delta(x_0) \cap \left(\Omega^+ \cup I\right).$$

Moreover, for $z \in B_\delta(x_0) \cap \Omega^-$,

$$g(z) = \sum_{n=0}^{\infty} \overline{b_n}(z - \overline{x_0})^n = \sum_{n=0}^{\infty} b_n (z - x_0)^n.$$

We conclude that

$$g(z) = \sum_{n=0}^{\infty} b_n (z - x_0)^n \quad \text{on } B_\delta(x_0).$$

Therefore g is also analytic on I, and hence analytic on Ω. Since $f = g$ on Ω^+, by the previous corollary, f is identical to g on Ω. That is, (4) holds. □