MATH2230B
Complex Variables with Applications

Lecturer: Chia-Yu Hsieh

Department of Mathematics
The Chinese University of Hong Kong

March 10, 2021

Theorem

Suppose that f *is analytic on an annulus* $B_{R_2}(z_0)\backslash\overline{B_{R_1}(z_0)}$. *Then* f *can be represented as*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \quad z \in B_{R_2}(z_0)\backslash\overline{B_{R_1}(z_0)}, \tag{1}$$

where

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}dw$$

with any simple closed curve γ *in* $B_{R_2}(z_0)\backslash\overline{B_{R_1}(z_0)}$ *around* z_0 *with counterclockwise orientation.*

Remark

(i) *The expression (1) is called the Laurent series of f about z_0.*

(ii) *If f is also analytic on $\overline{B_{R_1}(z_0)}$, then the Taylor series of f and the Laurent series of f about z_0 agree with each other. In fact, $c_n = 0$ for all $n < 0$ by Cauchy-Goursat theorem. Moreover, by Cauchy integral formula, $c_n = f^{(n)}(z_0)/n! = a_n$ for $n \in \mathbb{N} \cup \{0\}$, where a_n are the coefficients of Taylor series of f.*

(iii) *The coefficients of Laurent series are unique. Suppose that*

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=-\infty}^{\infty} \tilde{c}_n(z-z_0)^n \quad \text{on } B_{R_2}(z_0)\backslash\overline{B_{R_1}(z_0)}.$$

Then, for $m \in \mathbb{Z}$,

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^{n-m-1} = \sum_{n=-\infty}^{\infty} \tilde{c}_n(z-z_0)^{n-m-1}.$$

Remark (continued)

Since

$$\int_\gamma (z-z_0)^{n-m-1} = \begin{cases} 2\pi i & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

for any simple closed curve $\gamma \in B_{R_2}(z_0) \backslash \overline{B_{R_1}(z_0)}$ *around* z_0 *with counterclockwise orientation, then we conclude*

$$c_m = \tilde{c}_m, \quad m \in \mathbb{Z}.$$

Here the interchange of orders of summation and integration can be justified by uniform convergence.

Proof.

Without loss of generality, we may assume that $z_0 = 0$. For $z \in B_{R_2}(0) \backslash \overline{B_{R_1}(0)}$, let γ_1 and γ_2 be the circles centered at 0 with radius r_1 and r_2, respectively, such that $R_1 < r_1 < |z| < r_2 < R_2$. There is $\varepsilon > 0$ sufficiently small such that the closed disk $\overline{B_\varepsilon(z)}$ is contained in the annulus $B_{r_2}(0) \backslash \overline{B_{r_1}(0)}$. Let γ be the boundary of $B_\varepsilon(z)$. We assume that γ, γ_1 and γ_2 are all counterclockwise oriented. By Cauchy-Goursat theorem for multi-connected domains,

$$\int_{\gamma_2} \frac{f(w)}{w-z} dw - \int_{\gamma_1} \frac{f(w)}{w-z} dw - \int_{\gamma} \frac{f(w)}{w-z} dw = 0. \qquad (2)$$

By the Cauchy integral formula,

$$\int_{\gamma} \frac{f(w)}{w-z} dw = 2\pi i f(z).$$

Proof, continued.

Put this into (2), we obtain

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{z-w}dw. \tag{3}$$

Notice that

$$\frac{1}{w-z} = \frac{1}{w}\cdot\frac{1}{1-z/w} = \frac{1}{w}\left[\sum_{n=0}^{N}\left(\frac{z}{w}\right)^n + \frac{1}{1-z/w}\left(\frac{z}{w}\right)^{N+1}\right]$$

for $w \in \gamma_2$. Similarly,

$$\frac{1}{z-w} = \frac{1}{z}\left[\sum_{n=0}^{M}\left(\frac{w}{z}\right)^n + \frac{1}{1-w/z}\left(\frac{w}{z}\right)^{M+1}\right]$$

for $w \in \gamma_1$. Then (3) becomes

$$\begin{aligned} f(z) = \sum_{n=0}^{N}\frac{z^n}{2\pi i}\int_{\gamma_2}\frac{f(w)}{w^{n+1}}dw + \frac{z^{N+1}}{2\pi i}\int_{\gamma_2}\frac{f(w)}{(w-z)w^{N+1}}dw \\ + \sum_{n=0}^{M}\frac{z^{-(n+1)}}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w^{-n}}dw + \frac{z^{-(M+1)}}{2\pi i}\int_{\gamma_1}\frac{f(w)}{(z-w)w^{-(M+1)}}dw. \end{aligned}$$

Proof, continued.

Now, let $K = \max_{\gamma_1 \cup \gamma_2} |f|$. Then

$$\begin{aligned} &\left| \frac{z^{N+1}}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-z)w^{N+1}} dw \right| \\ &\leq \frac{|z|^{N+1}}{2\pi} \cdot \frac{K}{(r_2 - |z|){r_2}^{N+1}} \cdot 2\pi r_2 \\ &= K \cdot \frac{r_2}{r_2 - |z|} \left(\frac{|z|}{r_2} \right)^{N+1} \longrightarrow 0 \quad \text{as } N \to \infty, \end{aligned}$$

and

$$\begin{aligned} &\left| \frac{z^{-(M+1)}}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(z-w)w^{-(M+1)}} dw \right| \\ &\leq \frac{|z|^{-(M+1)}}{2\pi} \cdot \frac{K}{(|z| - r_1){r_1}^{-(M+1)}} \cdot 2\pi r_1 \\ &= K \cdot \frac{r_1}{|z| - r_1} \left(\frac{r_1}{|z|} \right)^{M+1} \longrightarrow 0 \quad \text{as } M \to \infty. \end{aligned}$$

Therefore, we complete the proof.

Example

Let $f(z) = \dfrac{1}{z(1+z^2)}$*. Since*

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \quad |z| < 1.$$

Therefore,

$$\frac{1}{z(1+z^2)} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1}, \quad 0 < |z| < 1,$$

is the Laurent series of f*.*

Example

Let $f(z) = \dfrac{z+1}{z-1}$. *For* $|z| < 1$,

$$\frac{z+1}{z-1} = -z \cdot \frac{1}{1-z} - \frac{1}{1-z} = -z \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} z^n$$
$$= -1 - 2 \sum_{n=1}^{\infty} z^n,$$

which is the Taylor series of f. *And for* $|z| > 1$,

$$\frac{z+1}{z-1} = \frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} = \left(1 + \frac{1}{z}\right) \sum_{n=0}^{\infty} \frac{1}{z^n} = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{z^n},$$

which is the Laurent series of f.

Example

Notice that

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!},$$

which is the Laurent series of $e^{1/z}$*. Let* γ *be the circle centered at* 0 *with radius* R*, counterclockwise oriented. Then the coefficients of Laurent series*

$$c_{-n} = \frac{1}{2\pi i} \int_\gamma \frac{e^{1/z}}{z^{-n+1}} dz = \frac{1}{n!}, \quad n \in \mathbb{N} \cup \{0\}.$$

It can be used to evaluate the integrals

$$\int_\gamma \frac{e^{1/z}}{z^{-n+1}} dz = \frac{2\pi i}{n!}, \quad n \in \mathbb{N} \cup \{0\}.$$