MATH2230B
Complex Variables with Applications

Lecturer: Chia-Yu Hsieh

Department of Mathematics
The Chinese University of Hong Kong

March 8, 2021

Definition

For $z_n \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, *the series* $\sum_{n=0}^{\infty} z_n$ *converges to the sum* z *if the partial sum*

$$\sum_{n=0}^{N} z_n \longrightarrow z \qquad \text{as } N \to \infty.$$

If it does not converge, we say that it diverges. And we say that the series $\sum_{n=0}^{\infty} z_n$ *converges absolutely if the series* $\sum_{n=0}^{\infty} |z_n|$ *converges.*

Proposition

Absolute convergence implies convergence.

Proposition

Given a power series $\sum_{n=0}^{\infty} a_n z^n$, *there exists* $0 \leq R \leq \infty$ *such that the series converges absolutely if* $|z| < R$ *and diverges if* $|z| > R$. *Moreover,* R *is given by*

$$R = \left(\limsup_{n \to \infty} |a_n|^{1/n} \right)^{-1}.$$

Definition

R *given in the last proposition is called the radius of convergence of the power series. And* $B_R(0)$ *is called the disc of convergence.*

Proof.

For $|z| < R$, there is $\varepsilon_1 > 0$ small enough such that

$$\left(R^{-1} + \varepsilon_1\right)|z| = r < 1.$$

By the definition of R,

$$|a_n|^{1/n} \leq R^{-1} + \varepsilon_1$$

for all n large, which gives

$$|a_n||z|^n \leq \left(R^{-1} + \varepsilon_1\right)^n |z|^n = r^n.$$

By a comparison with the series $\sum_{n=0}^{\infty} r^n$, the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

Proof, continued.

If $|z| > R$, there is $\varepsilon_2 > 0$ such that

$$\left(R^{-1} - \varepsilon_2\right)|z| > 1.$$

By the definition of R, there exist a subsequence, still denoted by a_n, such that

$$|a_n|^{1/n} \geq R^{-1} - \varepsilon_2.$$

We have

$$|a_n z^n| \geq \left(R^{-1} - \varepsilon_2\right)^n |z|^n \longrightarrow \infty \qquad \text{as } n \to \infty.$$

Thus, the series cannot converge for $|z| > R$. □

Theorem

A function defined by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C},$$

with positive radius of convergence, is differentiable on its disc of convergence. And its derivative can be represented by the power series

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

which has the same radius of convergence as f.

Proof.

Let

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Since

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |n a_n|^{1/n},$$

g has the same radius of convergence as f. Let R be the radius of convergence of f, and divide f into

$$f(z) = S_N(z) + R_N(z),$$

where

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \quad \text{and} \quad R_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Proof, continued.

For $|z_0| < r < R$, $|h|$ sufficiently small such that $|z_0 + h| < r$, we have

$$\begin{aligned} &\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) \\ &= \left(\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right) + \left(S_N'(z_0) - g(z_0)\right) \\ &\quad + \frac{R_N(z_0+h)-R_N(z_0)}{h} \end{aligned}$$

Given $\varepsilon > 0$, since

$$\begin{aligned} \left|\frac{R_N(z_0+h)-R_N(z_0)}{h}\right| &\leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}, \end{aligned}$$

Proof, continued.

there is $N_1 \in \mathbb{N}$ sufficiently large such that

$$\left|\frac{R_N(z_0+h) - R_N(z_0)}{h}\right| < \frac{\varepsilon}{3}$$

for all h with $|z_0 + h| < r$ and $N \geq N_1$. Also, since

$$\lim_{N\to\infty} S_N'(z_0) = g(z_0),$$

there is $N_2 \in \mathbb{N}$ sufficiently large such that

$$\left|S_N'(z_0) - g(z_0)\right| < \frac{\varepsilon}{3}$$

if $N \geq N_2$.

Proof, continued.

Now, we fix $N \geq \max\{N_1, N_2\}$, there is $\delta > 0$ such that

$$\left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S_N'(z_0) \right| < \frac{\varepsilon}{3}$$

provided $|h| < \delta$. Therefore,

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < \varepsilon$$

provided $|h| < \delta$, that is,

$$f'(z_0) = g(z_0).$$

□

Corollary

A function defined by a power series with positive radius of convergence is infinitely many times differentiable on its disc of convergence. And all the higher derivatives can be represented by the power series obtained by termwise differentiation and have the same radius of convergence as f.

Theorem

Suppose that f is analytic on a disc $B_R(z_0)$. Then f can be represented as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in B_R(z_0), \tag{1}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \in \mathbb{N} \cup \{0\}.$$

Remark

(i) *The expression (1) is called the Taylor series of f about z_0. In particular, if $z_0 = 0$, it is called the Maclaurin series of f.*

(ii) *The coefficients of Taylor series are unique.*

Proof.

Without loss of generality, we may assume that $z_0 = 0$. By Cauchy integral formula, for any $z \in B_R(0)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw, \tag{2}$$

where γ is the circle centered at 0 with radius $\dfrac{|z| + R}{2}$ and counterclockwise orientation. Notice that

$$\frac{1}{w - z} = \frac{1}{w} \cdot \frac{1}{1 - z/w} = \frac{1}{w} \left[\sum_{n=0}^{N} \left(\frac{z}{w} \right)^n + \frac{1}{1 - z/w} \left(\frac{z}{w} \right)^{N+1} \right].$$

Thus, (2) becomes

Proof, continued.

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w}\sum_{n=0}^{N}\left(\frac{z}{w}\right)^n dw \\
&\quad + \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w}\frac{1}{1-z/w}\left(\frac{z}{w}\right)^{N+1} dw \\
&= \sum_{n=0}^{N}\frac{z^n}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}dw + \frac{z^{N+1}}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)w^{N+1}}dw. \qquad (3)
\end{aligned}
$$

By the generalized Cauchy integral formula,

$$
\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}dw = \frac{f^{(n)}(0)}{n!}, \quad n = 0, 1, \dots, N.
$$

Proof, continued.

Thus, (3) reduces to

$$f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n + \frac{z^{N+1}}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)w^{N+1}} dw.$$

Now, let $M = \max_\gamma |f|$, then

$$\begin{aligned} &\left| \frac{z^{N+1}}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)w^{N+1}} dw \right| \\ \le\ & \frac{|z|^{N+1}}{2\pi} \cdot \frac{M}{\frac{R-|z|}{2}\left(\frac{R+|z|}{2}\right)^{N+1}} \cdot 2\pi \left(\frac{R+|z|}{2}\right) \\ =\ & M \cdot \frac{R+|z|}{R-|z|} \left(\frac{2|z|}{R+|z|}\right)^{N+1} \longrightarrow 0 \quad \text{as } N \to \infty. \end{aligned}$$

Therefore, we complete the proof. □

Example

Let $f(z) = \dfrac{1}{1-z}$*. We have*

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}, \quad z \neq 1, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{on } B_1(0).$$

As for the Taylor series of f *about* i*, we have*

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad \text{on } B_{\sqrt{2}}(i).$$

Example

Let $f(z) = e^z$. *We have*

$$f^{(n)}(z) = e^z \qquad \text{on } \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{on } \mathbb{C}.$$

We can use the Taylor series of e^z *to show that*

$$e^{2z} = \sum_{n=0}^{\infty} \frac{2^n z^n}{n!} \qquad \text{on } \mathbb{C}.$$

Moreover,

$$z^3 e^{2z} = \sum_{n=0}^{\infty} \frac{2^n z^{n+3}}{n!} \qquad \text{on } \mathbb{C}.$$

Example

Let $f(z) = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$. *We have*

$$f^{(n)}(z) = \frac{i^n e^{iz} - (-i)^n e^{-iz}}{2i} \quad \text{on } \mathbb{C}, \qquad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\begin{aligned}\sin z &= \sum_{n=0}^{\infty} \frac{i^n - (-i)^n}{2i} \cdot \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k+1)!} z^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \qquad \text{on } \mathbb{C}. \end{aligned} \tag{4}$$

Example

Let $f(z) = \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$. *We have*

$$f^{(n)}(z) = \frac{i^n e^{iz} + (-i)^n e^{-iz}}{2} \qquad \text{on } \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\begin{aligned}\cos z &= \sum_{n=0}^{\infty} \frac{i^n + (-i)^n}{2} \cdot \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!} z^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \qquad \text{on } \mathbb{C}.\end{aligned}$$

The Taylor series of $\cos z$ *can also be obtained by differentiating (4) term by term*

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot (2k+1) z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \qquad \text{on } \mathbb{C}.$$

Example

Let $f(z) = \sinh z = \dfrac{e^z - e^{-z}}{2}$. *We have*

$$f^{(n)}(z) = \frac{e^z - (-1)^n e^{-z}}{2} \quad on\ \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\sinh z = \sum_{n=0}^{\infty} \frac{1-(-1)^n}{2} \cdot \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \quad on\ \mathbb{C}.$$

Example

Let $f(z) = \cosh z = \dfrac{e^z + e^{-z}}{2}$. *We have*

$$f^{(n)}(z) = \frac{e^z + (-1)^n e^{-z}}{2} \quad \textit{on } \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\cosh z = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2} \cdot \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \quad \textit{on } \mathbb{C}.$$

Theorem

Suppose that f is analytic on an annulus $B_{R_2}(z_0) \backslash \overline{B_{R_1}(z_0)}$. Then f can be represented as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad z \in B_{R_2}(z_0) \backslash \overline{B_{R_1}(z_0)}, \tag{5}$$

where

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} dw$$

with any simple closed curve γ in $B_{R_2}(z_0) \backslash \overline{B_{R_1}(z_0)}$ around z_0 with counterclockwise orientation.