MATH2230B
Complex Variables with Applications

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Theorem (Liouville's theorem)

If f is entire and bounded, then f is constant.

Proof.

Since f is bounded, there is a positive constant M such that

$$|f(z)| \leq M$$

for all $z \in \mathbb{C}$. By Cauchy's inequality,

$$|f'(z)| \leq \frac{M}{R}$$

for any $z \in \mathbb{C}$ and $R > 0$. Since R is arbitrary, we obtain that

$$f'(z) = 0 \qquad \text{on } \mathbb{C}.$$

As a consequence, f is a constant on $\mathbb{C}$. □

Theorem (Fundamental theorem of algebra)

Any non-constant polynomial has at least one root.

Proof.

Suppose on the contrary that there is a polynomial

$$P(z) = a_0 + a_1 z + ... + a_n z^n, \quad n \geq 1, \quad a_n \neq 0,$$

such that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/P(z)$ is entire. Now, we claim that $1/P(z)$ is bounded. Let

$$w(z) = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + ... + \frac{a_{n-1}}{z}, \quad z \neq 0.$$

By the triangle inequality,

$$|w(z)| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + ... + \frac{|a_{n-1}|}{|z|}.$$

Proof, continued.

By choosing $R > 0$ sufficiently large, we have

$$\frac{|a_k|}{R^{n-k}} \leq \frac{|a_n|}{2n} \qquad \text{for } k = 0, 1, ..., n-1,$$

which gives

$$|w| \leq \frac{|a_n|}{2} \qquad \text{for all } |z| \geq R.$$

Consequently,

$$|a_n + w| \geq |a_n| - |w| \geq \frac{|a_n|}{2} \qquad \text{for all } |z| \geq R.$$

Proof, continued.

Then we have

$$|P(z)| = |a_n + w||z|^n \geq \frac{|a_n|}{2} R^n \qquad \text{for all } |z| \geq R,$$

and hence

$$\left|\frac{1}{P(z)}\right| \leq \frac{2}{|a_n| R^n} \qquad \text{for all } |z| \geq R.$$

Since $1/P(z)$ is continuous on the set $\{|z| \leq R\}$, it is bounded on $\{|z| \leq R\}$. Therefore, $1/P(z)$ is entire and bounded. By Liouville's theorem, $1/P(z)$ is a constant on $\mathbb{C}$, which leads a contradiction. $\square$

Corollary

A polynomial P of order n, $n \geq 1$ has precisely n roots in $\mathbb{C}$. P can be expressed as

$$P(z) = c(z - z_1)(z - z_2)...(z - z_n),$$

where c, z_1, ..., z_n are constants with $c \neq 0$.

Proof.

For

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n,$$

by the fundamental theorem of algebra, there is a root z_1 of P. We have

$$\begin{aligned} P(z) &= P((z - z_1) + z_1) \\ &= a_0 + a_1((z - z_1) + z_1) + \ldots + a_n((z - z_1) + z_1)^n \\ &= b_1(z - z_1) + b_2(z - z_1)^2 + \ldots + b_n(z - z_1)^n \end{aligned}$$

for some $b_1, \ldots, b_n \in \mathbb{C}$, and $b_n = a_n$. Thus,

$$\begin{aligned} P(z) &= (z - z_1)\left[b_1 + b_2(z - z_1) + \ldots + b_n(z - z_1)^{n-1}\right] \\ &= (z - z_1)Q(z), \end{aligned}$$

where Q is a polynomial of order $n - 1$. By the fundamental theorem of algebra again, there is a root z_2 of Q. We then prove the corollary inductively. □

Theorem (Maximum modulus principle)

If f is non-constant and analytic on an open connected set Ω, then there is no point $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$.

Lemma

If $|f(z)| \leq |f(z_0)|$ for all $z \in B_R(z_0)$, then $f(z) = f(z_0)$ for all $z \in B_R(z_0)$.

Proof of the lemma.

Let C_ρ be the circle centered at z_0 with radius $\rho \in (0, R)$ and counterclockwise oriented. By Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz$$

for all $\rho \in (0, R)$. Then

$$|f(z_0)| \leq \frac{1}{2\pi} \max_{C_\rho} \frac{|f(z)|}{|z - z_0|} \cdot 2\pi\rho \leq \frac{1}{2\pi} \cdot \frac{|f(z_0)|}{\rho} \cdot 2\pi\rho = |f(z_0)|$$

for all $\rho \in (0, R)$. Thus both the inequalities above are equalities, which implies that

$$|f(z)| = |f(z_0)| \qquad \text{on } C_\rho.$$

Since $\rho \in (0, R)$ is arbitrary, $|f(z)| = |f(z_0)|$ on $B_R(z_0)$. Since $|f|$ is a constant on $B_R(z_0)$, f is also a constant on $B_R(z_0)$, which completes the proof. □

Proof of maximum modulus principle.

Suppose on the contrary that there is $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$. For any $w \in \Omega$, there is a polygonal line L connecting z_0 and w. Let $0 < \delta < \mathrm{dist}(L, \partial\Omega)$, L can be covered by finitely many discs $B_\delta(z_k)$, $z_k \in L$, $k = 0, 1, ..., N$, and $w = z_N$. Moreover, $z_k \in B_\delta(z_{k-1})$ for each $k = 1, 2, ..., N$. By the lemma, f is a constant on $B_\delta(z_0)$. Thus $f(z_1) = f(z_0)$, and hence $|f(z)| \leq |f(z_1)|$ on $B_\delta(z_1)$. Continue in this manner, we conclude that $f(w) = f(z_0)$. That is, f is constant on Ω, which leads a contradiction. □

Remark

(i) *Under the assumptions of the maximum modulus principle, if f is continuous on the closure of Ω, then the maximum value of $|f(z)|$ on the closure of Ω must occur on the boundary of Ω.*

(ii) *Applying the maximum modulus principle to $1/f(z)$, the minimum of $|f(z)|$ cannot be obtained at an interior point of Ω provided that $f(z) \neq 0$ for all $z \in \Omega$.*

(iii) *Applying the maximum modulus principle to functions $e^{f(z)}$ and $e^{-if(z)}$, the maximum principle holds for the real and imaginary parts of f.*

Example

We can use the maximum modulus principle to prove the fundamental theorem of algebra. Suppose that P is a non-constant polynomial of order n and has no root on $\mathbb{C}$. Then $1/P(z)$ is analytic on $B_R(0)$ for all $R > 0$. As in the proof of the fundamental theorem of algebra, we have

$$\left|\frac{1}{P(z)}\right| \leq \frac{2}{|a_n|R^n} \qquad \text{on the circle } \{z : |z| = R\}$$

provided R sufficiently large. By the maximum modulus principle,

$$\left|\frac{1}{P(z)}\right| \leq \frac{2}{|a_n|R^n} \qquad \text{on } B_R(0).$$

Taking $R \to \infty$, we conclude that $\frac{1}{P(z)} = 0$ on $\mathbb{C}$, which is a contradiction.

Example

Let $f(z) = (z+1)^2$ *be defined on the closed triangle* T *with vertices* $z = 0$, $z = 2$ *and* $z = i$. *Notice that* $|f(z)|$ *can be interpreted as the square of the distance between* -1 *and* $z \in T$. *The maximum and minimum values of* $|f(z)|$ *occur at* $z = 2$ *and* $z = 0$, *respectively.*

Definition

For $z_n \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, *the series* $\sum_{n=0}^{\infty} z_n$ *converges to the sum* z *if the partial sum*

$$\sum_{n=0}^{N} z_n \longrightarrow z \qquad \text{as } N \to \infty.$$

If it does not converge, we say that it diverges. And we say that the series $\sum_{n=0}^{\infty} z_n$ *converges absolutely if the series* $\sum_{n=0}^{\infty} |z_n|$ *converges.*

Proposition

Absolute convergence implies convergence.