MATH2230B
Complex Variables with Applications

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Theorem (generalized Cauchy integral formula)

Let Ω is the open set enclosed by a simple closed curve γ with counterclockwise orientation. If f is analytic on some open set containing $\overline{\Omega}$, the closure of Ω, then f is differentiable of all orders in Ω. Moreover,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for all $z_0 \in \Omega$, $n \in \mathbb{N} \cup \{0\}$.

Proof.

The proof is by induction on n. $n = 0$ is the Cauchy integral formula we have proved. For $n \in \mathbb{N}$, suppose that f is $n-1$ times differentiable in Ω with

$$f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^n} dz$$

for all $z_0 \in \Omega$. Then for each $z_0 \in \Omega$, $|h|$ small enough such that $z_0 + h \in \Omega$, we have

$$\begin{aligned} &\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} \\ &= \frac{(n-1)!}{2\pi i} \int_\gamma f(z) \frac{1}{h} \left[\frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0)^n} \right] dz. \end{aligned}$$

Proof, continued.

Notice that

$$\begin{aligned}
&\frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0)^n} \\
&= \left[\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right] \sum_{k=0}^{n-1} \frac{1}{(z-z_0-h)^k (z-z_0)^{n-1-k}} \\
&= \frac{h}{(z-z_0-h)(z-z_0)} \sum_{k=0}^{n-1} \frac{1}{(z-z_0-h)^k (z-z_0)^{n-1-k}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} \\
&= \frac{(n-1)!}{2\pi i} \int_\gamma f(z) \frac{1}{(z-z_0)^2} \frac{n}{(z-z_0)^{n-1}} dz \\
&= \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz,
\end{aligned}$$

which completes the proof. □

Example

Let $f(z) = e^{2z}$. *To evaluate the integral*

$$\int_\gamma \frac{e^{2z}}{z^4}\,dz,$$

where γ *is the unit circle centered at the origin with counterclockwise orientation, we have*

$$\int_\gamma \frac{e^{2z}}{z^4}\,dz = \int_\gamma \frac{f(z)}{(z-0)^4}\,dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{8\pi i}{3}.$$

Corollary

If f is differentiable on an open set Ω, then f is differentiable of all orders on Ω. As a consequence, the real and imaginary parts of f are continuously differentiable of all orders.

Proof.

Given $z_0 \in \Omega$, there is a disc B centered at z_0 such that the closure of B is contained in Ω. Then we can apply the theorem for the generalized Cauchy integral formula to conclude that f is differentiable of all orders in B. □

Theorem (Morera's theorem)

Let f is continuous on an open connected set Ω. If

$$\int_\gamma f(z)dz = 0$$

for all closed curves in Ω, then f is differentiable on Ω.

Proof.

The assumptions imply that f has an antiderivative F. By the last corollary, F is differentiable of all orders on Ω. Hence, so is f. □

Corollary (Cauchy's inequality)

Suppose f is differentiable on an open set containing the closure of a disc B centered at z_0 with radius R. Let γ be the circle centered at z_0 with radius R, counterclockwise oriented, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! \max_\gamma |f(z)|}{R^n}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Proof.

By Cauchy integral formula,

$$\begin{aligned}\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \\ &\leq \frac{n!}{2\pi}\cdot\frac{\max_\gamma |f(z)|}{R^{n+1}}\cdot 2\pi R \\ &= \frac{n! \max_\gamma |f(z)|}{R^n}.\end{aligned}$$

□