MATH2230B
Complex Variables with Applications

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Theorem (Cauchy-Goursat theorem)

If f is analytic at all points interior to and on a simple closed curve γ, then

$$\int_{\gamma} f(z)dz = 0.$$

Theorem (Cauchy-Goursat theorem for rectangles)

If f is differentiable on an open set Ω, and γ is the boundary of a rectangle contained in Ω, then

$$\int_\gamma f(z)dz = 0.$$

Theorem

If f is differentiable on an open disc Ω, then

$$\int_\gamma f(z)dz = 0$$

for all closed curve γ in Ω.

Proof of Cauchy-Goursat theorem.

Let K be the closed region bounded by γ. By the assumption, f is differentiable on some bounded open set Ω containing K, and we have $\operatorname{dist}(K, \partial\Omega) > 3\varepsilon$ for some $\varepsilon > 0$, where $\partial\Omega$ is the boundary of Ω. We may assume $\gamma_0 = \gamma$ is counterclockwise oriented. And let γ_1 be a simple closed curve lying in the interior of K, also counterclockwise oriented, such that $\operatorname{dist}(z, \gamma_0) < \varepsilon$ for every $z \in \gamma_1$. And then we slice the strip bounded by γ_0 and γ_1 into small pieces. Each piece is contained in a disc with radius 2ε contained in Ω. By the previous theorem, summing the integrals on the boundaries of all the pieces, we obtain

$$\int_{\gamma_0} f(z)dz + \int_{-\gamma_1} f(z)dz = 0.$$

Here the integrals on common edges are cancelled out. The above equality shows that

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Proof of Cauchy-Goursat theorem, continued.

We can continue in this manner to obtain a sequence of simple closed curves γ_0, γ_1, γ_2, ..., γ_n, ..., with $\mathrm{length}(\gamma_n) \to 0$ as $n \to \infty$, and

$$\int_{\gamma_n} f(z)dz = \int_{\gamma_{n+1}} f(z)dz$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore,

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_n} f(z)dz$$

for all $n \in \mathbb{N}$. In addition, for γ_n, we have

$$\left|\int_{\gamma_n} f(z)dz\right| \leq \max_K |f| \cdot \mathrm{length}(\gamma_n) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus, we complete the proof. □

Definition

A connected set Ω is called simply connected if every simple closed curve in Ω encloses only points in Ω. If a connected set Ω is not simply connected, then it is called multiply connected.

Theorem

If f is analytic on an open simply connected domain Ω, then

$$\int_\gamma f(z)dz = 0$$

for all closed curve γ lying in Ω.

Corollary

If f is analytic on an open simply connected domain, then f has an antiderivative. And the integral of f from one point to another is independent of paths.

Example

Let γ be any closed curve lying in the disc $B_2(0)$. Then

$$\int_\gamma \frac{\sin z}{(z^2+9)^5}\,dz = 0.$$

Example

Let Ω be an open simply connected set with $1 \in \Omega$, $0 \notin \Omega$. Then there is a branch of the logarithm f on Ω such that

$$f(x) = \ln x \quad \text{for } x \in \mathbb{R}, \quad x \text{ near } 1.$$

It can be done by defining

$$f(z) = \int_\gamma \frac{dw}{w}$$

for any curve γ in Ω from 1 to z.

Example (continued)

Notice that the integral of $\frac{1}{z}$ *from* 1 *to* z *is independent of the choice of* γ *by the previous corollary. A similar argument in the proof of the theorem in Lecture 9, page 6, gives* $f'(z) = \frac{1}{z}$ *on* Ω*, and hence*

$$\left(ze^{-f(z)}\right)' = 0 \quad \text{on } \Omega.$$

Therefore, $ze^{-f(z)}$ *is a constant. By taking the value at* $z = 1$*, we conclude that* $ze^{-f(z)} \equiv 1$*, that is,*

$$e^{f(z)} = z \quad \text{on } \Omega.$$

As for $x \in \mathbb{R}$ *near* 1*, we have*

$$f(x) = \int_1^x \frac{dy}{y} = \ln x.$$

Theorem

Let γ_0*,* γ_1*, ...,* γ_n *be simple closed curves with counterclockwise orientation.* γ_k*'s,* $k = 1, ..., n$*, lying in the interior of* γ_0*, are disjoint, whose interiors have no points in common. If* f *is analytic on all the curves and throughout the multiply connected domain consisting of the points inside* γ_0 *and exterior to each* γ_k*,* $k = 1, ..., n$*, then*

$$\int_{\gamma_0} f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz.$$

Proof.

The theorem follows by dividing the domain into finitely many simply connected domains. □

Corollary

Let γ_1 and γ_2 be two simple closed curves with counterclockwise orientation. And γ_1 lies in the interior enclosed by γ_2. If f is analytic on the closed set consisting of γ_1, γ_2 and all points between them, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Example

Let γ be a simple closed curve with counterclockwise orientation surrounding the origin. We are going to evaluate the integral

$$\int_\gamma \frac{dz}{z}.$$

By using the definition of contour integrals, one can see that

$$\int_C \frac{dz}{z} = 2\pi i$$

for any circle C centered at the origin with counterclockwise orientation. Thus, by the previous corollary,

$$\int_\gamma \frac{dz}{z} = 2\pi i.$$

Theorem (Cauchy integral formula)

Let Ω be the open set enclosed by a simple closed curve γ with counterclockwise orientation. If f is analytic on $\overline{\Omega}$, the closure of Ω, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz$$

for all $z_0 \in \Omega$.

Proof.

For $z_0 \in \Omega$, let C_ρ be the circle centered at z_0 with radius ρ sufficiently small such that $C_\rho \subset \Omega$. We assume that C_ρ is counterclockwise oriented. By the previous corollary,

$$\int_\gamma \frac{f(z)}{z - z_0} dz = \int_{C_\rho} \frac{f(z)}{z - z_0} dz.$$

Therefore, we have

$$\int_\gamma \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_\rho} \frac{1}{z - z_0} dz = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz. \tag{1}$$

Also, we recall that,

$$\int_{C_\rho} \frac{1}{z - z_0} dz = 2\pi i.$$

Proof, continued.

Then (1) becomes

$$\int_\gamma \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}dz.$$

For the right-hand side on the last equality, we have

$$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \leq \sup_{z\in\overline{\Omega}\setminus\{z_0\}} \left|\frac{f(z)-f(z_0)}{z-z_0}\right| \cdot 2\pi\rho \longrightarrow 0$$

as $\rho \to 0$. Here $\sup_{z\in\overline{\Omega}\setminus\{z_0\}} \left|\frac{f(z)-f(z_0)}{z-z_0}\right|$ is finite since f is analytic on $\overline{\Omega}$. We conclude that

$$\int_\gamma \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = 0.$$

The theorem then follows. □

Example

Let $f(z) = \frac{\cos z}{z^2 + 9}$*. To evaluate the integral*

$$\int_\gamma \frac{\cos z}{z\,(z^2 + 9)}\,dz,$$

where γ *is the unit circle centered at the origin with counterclockwise orientation, we have*

$$\int_\gamma \frac{\cos z}{z\,(z^2 + 9)}\,dz = \int_\gamma \frac{f(z)}{z - 0}\,dz = 2\pi i f(0) = \frac{2\pi i}{9}.$$

Theorem (generalized Cauchy integral formula)

Let Ω is the open set enclosed by a simple closed curve γ with counterclockwise orientation. If f is analytic on some open set containing $\overline{\Omega}$, the closure of Ω, then f is differentiable of all orders in Ω. Moreover,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for all $z_0 \in \Omega$, $n \in \mathbb{N} \cup \{0\}$.