MATH2230B
Complex Variables with Applications

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Theorem (Cauchy-Goursat theorem)

If f is analytic at all points interior to and on a simple closed curve γ, then

$$\int_\gamma f(z)dz = 0.$$

Theorem (Cauchy-Goursat theorem for rectangles)

If f is differentiable on an open set Ω, and γ is the boundary of a rectangle contained in Ω, then

$$\int_\gamma f(z)dz = 0.$$

Proof.

Let R_0 be the closed rectangle with boundary γ. Assume that $\gamma_0 = \gamma = l_1 \cup l_2 \cup l_3 \cup l_4$, counterclockwise oriented. Let z_k be the midpoint of l_k, $k = 1, ..., 4$. By connecting z_1 and z_3, and connecting z_2 and z_4, we obtain four smaller rectangles with boundaries $\gamma_{1,1}$, $\gamma_{1,2}$, $\gamma_{1,3}$ and $\gamma_{1,4}$. We assume that $\gamma_{1,j}$, $j = 1, ..., 4$, are all counterclockwise oriented. We have

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_{1,1}} f(z)dz + \int_{\gamma_{1,2}} f(z)dz + \int_{\gamma_{1,3}} f(z)dz + \int_{\gamma_{1,4}} f(z)dz.$$

By the triangle inequality,

$$\left|\int_{\gamma_0} f(z)dz\right| \leq \left|\int_{\gamma_{1,1}} f(z)dz\right| + \left|\int_{\gamma_{1,2}} f(z)dz\right| + \left|\int_{\gamma_{1,3}} f(z)dz\right| + \left|\int_{\gamma_{1,4}} f(z)dz\right|.$$

Proof, continued.

There must be a $j \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{\gamma_0} f(z)dz \right| \leq 4 \left| \int_{\gamma_{1,j}} f(z)dz \right|.$$

Let $\gamma_1 = \gamma_{1,j}$ with j such that the last inequality holds, and R_1 be the closed rectangle with boundary γ_1. We can repeat the same process. For γ_n given, we divide the rectangle into four parts with boundary $\gamma_{n+1,j}$, $j = 1, ..., 4$. And we can choose a j such that

$$\left| \int_{\gamma_n} f(z)dz \right| \leq 4 \left| \int_{\gamma_{n+1,j}} f(z)dz \right|.$$

And then we denote $\gamma_{n+1} = \gamma_{n+1,j}$ with j such that the last inequality holds.

Proof, continued.

We obtain a sequence of rectangles R_n with boundaries γ_n, $n \in \mathbb{N} \cup \{0\}$, such that

$$R_0 \supset R_1 \supset \ldots \supset R_n \supset \ldots \tag{1}$$

and

$$\left| \int_{\gamma_0} f(z)dz \right| \leq 4^n \left| \int_{\gamma_n} f(z)dz \right|. \tag{2}$$

Since R_n's are compact satisfying (1) with $\mathrm{diam}(R_n) \to 0$ as $n \to \infty$, there is a unique $z_0 \in \Omega$ such that $z_0 \in R_n$ for all n. Since f is differentiable at z_0,

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| = 0.$$

Proof, continued.

On each γ_n, since constants and polynomials have antiderivatives,

$$\int_{\gamma_n} f(z)dz = \int_{\gamma_n} \left[f(z) - f(z_0) - f'(z_0)(z - z_0)\right] dz$$
$$= \int_{\gamma_n} \left[\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right] (z - z_0)dz.$$

Therefore,

$$\left|\int_{\gamma_n} f(z)dz\right|$$
$$\leq \sup_{z\in\gamma_n} \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \sup_{z\in\gamma_n} |z - z_0| \cdot \mathrm{length}(\gamma_n).$$

Notice that

$$\sup_{z\in\gamma_n} |z - z_0| \leq 2^{-n}L,$$

Proof, continued.

where L is the length of the diagonal of R_0, and

$$\mathrm{length}(\gamma_n) = 2^{-n}\mathrm{length}(\gamma).$$

Therefore,

$$\left|\int_{\gamma_n} f(z)dz\right| \leq 4^{-n}L \cdot \mathrm{length}(\gamma) \sup_{z\in\gamma_n}\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right|. \quad (3)$$

Combining (2) and (3), we obtain

$$\left|\int_{\gamma} f(z)dz\right| \leq L \cdot \mathrm{length}(\gamma) \sup_{z\in\gamma_n}\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \longrightarrow 0$$

as $n \to \infty$. □

Theorem

If f is differentiable on an open disc Ω, then

$$\int_\gamma f(z)dz = 0$$

for all closed curve γ in Ω.

Proof.

Without loss of generality, we may assume that the disc is centered at 0. Define

$$F(z) = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz,$$

where γ_1 is the line segment from 0 to $\mathrm{Re}\, z$, and γ_2 is the line segment from $\mathrm{Re}\, z$ to z. By Cauchy-Goursat theorem for rectangles, for $h = h_1 + h_2 i$ with $|h|$ sufficiently small,

$$F(z+h) - F(z) = \int_{\gamma} f(z)dz,$$

where γ is the polygonal line starting from z to $z + h_1$ and then from $z + h_1$ to $z + h$.

Proof, continued.

We have

$$\begin{aligned}
&\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| \\
&=\left|\frac{1}{h}\left[\int_0^1 (f(z+h_1t)-f(z))\,h_1dt\right.\right. \\
&\qquad \left.\left.+\int_0^1 (f(z+h_1+ih_2s)-f(z))\,ih_2ds\right]\right| \\
&\leq \sup_{t\in[0,1],\, s\in[0,1]} \left[|f(z+h_1t)-f(z)|+|f(z+h_1+ih_2s)-f(z)|\right] \\
&\longrightarrow 0 \quad \text{as } h\to 0.
\end{aligned}$$

That is, F is an antiderivative of f, and hence the theorem follows. □