MATH2230B
Complex Variables with Applications

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Definition

Let f be a function defined on an open connected set Ω. If there is a differentiable function F such that $F' = f$ on Ω, then we call F an antiderivative of f.

Remark

Antiderivatives of a given function are unique up to a constant.

Theorem

Let f be a continuous function on an open connected set Ω. If f has an antiderivative F on Ω, then for any piecewise smooth curve γ from z_1 to z_2 for some $z_1, z_2 \in \Omega$, we have

$$\int_\gamma f(z)dz = F(z_2) - F(z_1).$$

Remark

In particular, if f has an antiderivative, then the integral of f along any piecewise smooth closed curve equals to 0.

Proof.

Let γ be parametrized by $z(t) : [a, b] \to \Omega$.
Case 1: γ is smooth:
Suppose that $F = U + iV$, $z(t) = x(t) + iy(t)$, by using the Cauchy-Riemann equations, we have

$$\begin{aligned}
&\frac{d}{dt}F(z(t)) = \frac{d}{dt}F(x(t)+iy(t)) \\
&= \frac{d}{dt}U(x(t),y(t)) + i\frac{d}{dt}V(x(t),y(t)) \\
&= U_x(x(t),y(t))x'(t) + U_y(x(t),y(t))y'(t) \\
&\qquad + iV_x(x(t),y(t))x'(t) + iV_y(x(t),y(t))y'(t) \\
&= U_x(x(t),y(t))x'(t) - V_x(x(t),y(t))y'(t) \\
&\qquad + iV_x(x(t),y(t))x'(t) + iU_x(x(t),y(t))y'(t) \\
&= (U_x(x(t),y(t)) + iV_x(x(t),y(t)))(x'(t) + iy'(t)) \\
&= f(z(t))z'(t).
\end{aligned}$$

Proof, continued.

Therefore,

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b \frac{d}{dt}F(z(t))dt$$
$$= F(z(b)) - F(z(a)) = F(z_2) - F(z_1).$$

Case 2: γ is only piecewise smooth:
Let z be smooth on each interval $[a_{k-1}, a_k]$, $k = 1, ..., n$, where $a = a_0 < a_1 < ... < a_n = b$. Then

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} [F(z(a_k)) - F(z(a_{k-1}))]$$
$$= F(z(b)) - F(z(a)) = F(z_2) - F(z_1).$$

□

Theorem

Let f be a continuous function on an open connected set Ω. If

$$\int_{\gamma} f(z)dz = 0$$

for all piecewise smooth closed curve γ in Ω, then f has an antiderivative.

Lemma

Under the same assumption as in the above theorem, given $z_1, z_2 \in \Omega$,

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

for any piecewise smooth curves γ_1 *and* γ_2 *from* z_1 *to* z_2.

Proof of the lemma.

Let γ_1 and γ_2 be two piecewise smooth curves from z_1 to z_2, $z_1, z_2 \in \Omega$, we have

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \int_{\gamma_1 \cup (-\gamma_2)} f(z)dz = 0.$$

□

Proof of the theorem.

Fix $z_0 \in \Omega$. In view of the lemma, we can define a function

$$F(z) = \int_{\gamma_{z_0,z}} f(w)dw, \quad z \in \Omega,$$

where $\gamma_{z_0,z}$ is any smooth curve from z_0 to z. Then, for each $z \in \Omega$ and $h \in \mathbb{C}$ with $|h|$ sufficiently small,

$$\begin{aligned} F(z+h) - F(z) &= \int_{\gamma_{z_0,z+h}} f(w)dw - \int_{\gamma_{z_0,z}} f(w)dw \\ &= \int_{\gamma_{z,z+h}} f(w)dw, \end{aligned}$$

where $\gamma_{z,z'}$ denotes a curve lying in Ω from z to z'.

Proof of the theorem, continued.

Since the integration is independent of the choice of curves, we have

$$F(z+h)-F(z)=\int_0^1 f(z+ht)h\,dt,$$

and hence

$$\begin{aligned}\frac{F(z+h)-F(z)}{h}-f(z)&=\frac{1}{h}\int_0^1 [f(z+ht)-f(z)]\,h\,dt\\ &=\int_0^1 [f(z+ht)-f(z)]\,dt.\end{aligned}$$

Notice that by the continuity of f,

$$\left|\int_0^1 [f(z+ht)-f(z)]\,dt\right|\leq \sup_{t\in[0,1]}|f(z+ht)-f(z)|\to 0$$

as $h\to 0$, which implies

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z).$$

□

Remark

To summarize, the following three statements are equivalent for a continuous function f.

(i) *f has an antiderivative.*

(ii) *Integration of f from one point to another is independent of the choice of curves.*

(iii) *Integrals of f along closed curves have value 0.*

Example

The continuous function $f(z) = e^{\pi z}$ has an antiderivative $F(z) = e^{\pi z}/\pi$ on $\mathbb{C}$. Hence, for any piecewise smooth curve γ from i to $i/2$, we have

$$\int_\gamma e^{\pi z} dz = \frac{e^{\pi z}}{\pi}\bigg|_i^{i/2} = \frac{1+i}{\pi}.$$

Example

The function $f(z) = 1/z^2$ *has an antiderivative* $F(z) = -1/z$ *on* $\mathbb{C}\backslash\{0\}$*. Hence,*

$$\int_\gamma \frac{dz}{z^2} = 0,$$

where γ *is the unit circle parametrized by* $z(\theta) = e^{i\theta}$, $\theta \in [-\pi, \pi]$*. As for the function* $g(z) = 1/z$*, the integral of* g *along* γ *cannot be evaluated in a similar way. Notice that given a branch of the logarithm,* $G(z) = \log z$ *is an antiderivative of* $1/z$ *on the domain where the logarithm is defined. But the domain of* G *cannot contain the whole curve* γ*.*

Example

To evaluate the integral

$$\int_\gamma \frac{dz}{z},$$

where γ *is defined as in the last example, we can divide* γ *into two parts:* γ_1 *is the right half from* $-i$ *to* i *parametrized by* $z_1(\theta) = e^{i\theta}$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *and* γ_2 *is the left half from* i *to* $-i$ *parametrized by* $z_2(\theta) = e^{i\theta}$, $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. *For* γ_1, *we know that the principal branch of the logarithm is an antiderivative of* $1/z$ *on an open set containing* γ_1. *Thus,*

$$\int_{\gamma_1} \frac{dz}{z} = \log z\Big|_{-i}^{i} = \log i - \log(-i) = \frac{\pi i}{2} - \left(-\frac{\pi i}{2}\right) = \pi i,$$

where we used the principal branch of the logarithm here.

Example (continued)

As for γ_2, by using the branch of the logarithm

$$\log z = \ln|z| + i\theta, \quad \textit{where } \theta \in \arg z, \quad \theta \in (0, 2\pi),$$

defined on $\{|z| > 0, \mathrm{Arg}\, z \neq 0\}$, we have

$$\int_{\gamma_2} \frac{dz}{z} = \log z\Big|_{i}^{-i} = \log(-i) - \log i = \frac{3\pi i}{2} - \frac{\pi i}{2} = \pi i.$$

Therefore,

$$\int_{\gamma} \frac{dz}{z} = \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} = 2\pi i.$$

Example

Let f *be the square-root function on* $\left\{|z|>0, \operatorname{Arg} z \neq -\frac{\pi}{2}\right\}$ *defined by*

$$f(z)=z^{1/2}=e^{\frac{1}{2}\log z}=|z|^{1/2}e^{i\theta/2} \quad \textit{if } z=|z|e^{i\theta}, \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

That is, the power function is defined by using the following branch of the logarithm

$$\log z = \ln|z| + i\theta, \quad \textit{where } \theta \in \arg z, \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

If γ *is a curve from* -3 *to* 3 *lying above the real axis except for the endpoints, noticing that*

$$\left(z^{3/2}\right)' = \frac{3}{2}z^{1/2},$$

we have

$$\int_\gamma f(z)dz = \frac{2}{3}z^{3/2}\Big|_{-3}^{3} = 2\sqrt{3}\,(1+i)\,.$$

Theorem (Cauchy-Goursat theorem)

If f is analytic at all points interior to and on a simple closed curve γ, then

$$\int_\gamma f(z)dz = 0.$$

Theorem (Cauchy-Goursat theorem for rectangles)

If f is differentiable on an open set Ω, and γ is the boundary of a rectangle contained in Ω, then

$$\int_\gamma f(z)dz = 0.$$