MATH2230B
Complex Variables with Applications

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Example

Let γ *be the semicircular path parametrized by*

$$z(\theta) = 3e^{i\theta}, \quad \theta \in [0, \pi],$$

and $f(z) = z^{1/2}$ *be defined by using the branch of the logarithm*

$$\log z = \ln|z| + i\theta, \quad \theta \in \arg z, \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

Then

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^\pi \left(3e^{i\theta}\right)^{1/2} 3ie^{i\theta} d\theta = \int_0^\pi e^{\frac{1}{2}(\ln 3 + i\theta)} 3ie^{i\theta} d\theta \\ &= 3\sqrt{3}i \int_0^\pi e^{3i\theta/2} d\theta = 2\sqrt{3}\left(e^{3\pi i/2} - 1\right) \\ &= -2\sqrt{3}\,(1+i)\,.\end{aligned}$$

Example

Let γ *be the unit circle with parametrization*

$$z(\theta) = e^{i\theta}, \quad \theta \in [-\pi, \pi].$$

And let $f(z) = z^{-1+i}$ *be defined by using the principal branch of the logarithm. Notice that* f *is defined only for* $\theta \in (-\pi, \pi)$ *on* γ*. On the other hand, for* $\theta \in (-\pi, \pi)$*,*

$$f(z(\theta))z'(\theta) = e^{(-1+i)(i\theta)} ie^{i\theta} = ie^{-\theta}$$

is continuous in $(-\pi, \pi)$ *and has limits at* $\theta = \pm\pi$*. Thus, the (improper) integral exists and*

$$\int_\gamma f(z)dz = \int_{-\pi}^{\pi} ie^{-\theta} d\theta = i\left(e^{\pi} - e^{-\pi}\right).$$

Example

To estimate $\left|\int_\gamma \frac{z-2}{z^4+1}dz\right|$, *where* γ *is the arc of the circle* $|z|=2$ *from* $z=2$ *to* $z=2i$, *we have*

$$\left|\frac{z-2}{z^4+1}\right| \le \frac{|z|+2}{|z|^4-1} = \frac{4}{15} \quad \textit{for } |z|=2.$$

Therefore,

$$\left|\int_\gamma \frac{z-2}{z^4+1}dz\right| \le \frac{4}{15}\,\mathrm{length}(\gamma) = \frac{4\pi}{15}.$$

Example

Let γ_R be the semicircle parametrized by

$$z(\theta) = Re^{i\theta}, \quad \theta \in [0, \pi].$$

We are going to show that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{z+1}{(z^2+4)(z^2+9)} dz = 0$$

without actually evaluating the integral. Notice that

$$\begin{aligned} \left| \frac{z+1}{(z^2+4)(z^2+9)} \right| &\leq \frac{|z|+1}{(|z|^2-4)(|z|^2-9)} \\ &= \frac{R+1}{(R^2-4)(R^2-9)} \quad \textit{on } \gamma_R, \quad R > 3. \end{aligned}$$

Example (continued)

Thus, for $R > 3$,

$$\left| \int_{\gamma_R} \frac{z+1}{(z^2+4)(z^2+9)} dz \right| \leq \frac{R+1}{(R^2-4)(R^2-9)} \cdot \pi R \longrightarrow 0$$

as $R \to \infty$. *As a consequence, we obtain the limit*

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{z+1}{(z^2+4)(z^2+9)} dz = 0.$$