MATH2230B
Complex Variables with Applications

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February 1, 2021

Theorem

If $f'(z) = 0$ on an open connected set Ω, then f is a constant on Ω.

Lemma

If an open set Ω is connected, then it is polygonally connected. That is, for any $z_1, z_2 \in \Omega$, z_1 and z_2 can be connected by a polygonal line consisting of finitely many line segments in Ω.

Proof of the lemma.

If $\Omega = \phi$, then there is nothing to prove. By choosing a point $z_0 \in \Omega$, we define the set

$$S = \{z \in \Omega : z \text{ can be connected to } z_0 \text{ by a polygonal line}\}.$$

Given a point $z_1 \in S$, since Ω is open, there is $\varepsilon_1 > 0$ small enough such that $B_{\varepsilon_1}(z_1) \subset \Omega$. Notice that any point in $B_{\varepsilon_1}(z_1)$ can be connected to z_1 by a line segment. Thus, $B_{\varepsilon_1}(z_1) \subset S$, which implies that S is open.

Suppose that $\Omega \backslash S \neq \phi$, say, there is $z_2 \in \Omega \backslash S$. Again, we have $B_{\varepsilon_2}(z_2) \subset S$ for some $\varepsilon_2 > 0$. All point in $B_{\varepsilon_2}(z_2)$ do not belong to S. Otherwise, z_2 can be polygonally connected to z_0. Thus, $\Omega \backslash S$ is also open, which leads a contradiction. We conclude that $\Omega \backslash S = \phi$, i.e., $S = \Omega$. Therefore, for any two points $w_1, w_2 \in \Omega$, they can be connected by a polygonal line in Ω by combining one polygonal line connecting z_0 to w_1 and another one connecting z_0 to w_2. □

Proof of the theorem.

Let $f(z) = u(x, y) + iv(x, y)$ for $z = x + yi$. Since $f'(z) = 0$,

$$f'(z) = u_x(x, y) + iv_x(x, y) = 0.$$

In view of the Cauchy-Riemann equations, we have

$$u_x = u_y = v_x = v_y = 0 \quad \text{on } \Omega.$$

Next, if $z_1, z_2 \in \Omega$ such that the line segment L between z_1 and z_2 lie in Ω, we will show that $f(z)$ is a constant on L. L can be parametrized by

$$L = \left\{ z_1 + sw : s \in \left[0, |z_2 - z_1|\right] \right\},$$

where $w = w_1 + w_2 i = \dfrac{z_2 - z_1}{|z_2 - z_1|}$ is the unit vector in the direction from z_1 to z_2. Now, we consider the restriction of u on L, i.e., $u(x_1 + w_1 s, y_1 + w_2 s)$, where $z_1 = x_1 + y_1 i$.

Proof of the theorem, continued.

We have

$$\frac{d}{ds}u(x_1 + w_1 s, y_1 + w_2 s) = \nabla u\Big|_{(x_1+w_1 s, y_1+w_2 s)} \cdot (w_1, w_2),$$

where $\nabla u = (u_x, u_y)$ is the gradient of u. Since $u_x = u_y = 0$, it follows that

$$\frac{d}{ds}u(x_1 + w_1 s, y_1 + w_2 s) = 0 \quad \text{on } \big[0, |z_2 - z_1|\big].$$

This gives u is a constant on L. Since there is always a finite number of line segments connecting any two points in Ω, u is a constant on Ω. Similarly, by applying the same arguments to v, v is a constant on Ω. Therefore, f is a constant on Ω. □

Example

Suppose that f *and* $\overline{f}$ *are both differentiable on an open connected set* Ω*, we are going to show that* f *must be a constant.*
By writing $f(z) = u(x, y) + iv(x, y)$*,* $z = x + yi$*, we have* $\overline{f(z)} = u(x, y) - iv(x, y)$*. Since* f *is differentiable on* Ω*, the Cauchy-Riemann equations*

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{hold on } \Omega.$$

Since $\overline{f}$ *is also differentiable on* Ω*, the Cauchy-Riemann equations*

$$u_x = -v_y \quad \text{and} \quad u_y = v_x \quad \text{hold on } \Omega.$$

Therefore, we have $u_x = u_y = v_x = v_y = 0$ *on* Ω*, which implies* $f'(z) = 0$ *on* Ω*. Therefore,* f *is a constant.*

Example

Suppose that f is differentiable on an open connected set Ω. If $|f|$ is a constant on Ω, we are going to that f must be a constant. If $|f| = 0$ on Ω, then it follows that $f = 0$ on Ω. Now, we assume that $|f| = c \neq 0$ on Ω, we have

$$f(z)\overline{f(z)} = |f|^2 = c^2 \neq 0.$$

Notice that $f \neq 0$ on Ω. And hence

$$\overline{f(z)} = \frac{c^2}{f(z)}$$

is differentiable on Ω. The last example implies that f is a constant.

Theorem

Suppost that $f = u + iv$ is analytic on an open set Ω. Then u and v are harmonic functions on Ω.

Proof.

To show this, we need to use the fact that if a complex function is analytic at a point, then its real and imaginary parts have continuous partial derivatives of all orders there. Since u and v satisfy the Cauchy-Riemann equations, it holds that

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{on } \Omega.$$

Therefore,

$$u_{xx} = v_{xy} \quad \text{and} \quad u_{yy} = -v_{xy} \quad \text{on } \Omega.$$

We get

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0 \quad \text{on } \Omega.$$

That is, u is a harmonic function on Ω. The arguments for v is similar. □

Definition

Let w be a complex-valued function of a real variable t, written as

$$w(t) = u(t) + iv(t),$$

for some real-valued functions u and v. The derivative of w is defined by

$$\frac{d}{dt}w(t) = w'(t) = u'(t) + iv'(t),$$

provided that u and v are differentiable. And the definite integral of w over an interval $[a, b]$ is defined by

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the integrals on the right-hand side exist.

Example

$$\begin{aligned}\int_0^{\pi/4} e^{it}\,dt &= \int_0^{\pi/4} (\cos t + i \sin t)\, dt \\ &= \int_0^{\pi/4} \cos t dt + i \int_0^{\pi/4} \sin t dt \\ &= \sin t \Big|_0^{\pi/4} + i(-\cos t)\Big|_0^{\pi/4} \\ &= \frac{\sqrt{2}}{2} + \left(1 - \frac{\sqrt{2}}{2}\right) i.\end{aligned}$$

Proposition

If $w(t) = u(t) + iv(t)$ *is a complex-valued function on* $[a, b]$, *and* $W'(t) = w(t)$, *i.e.,* $W(t) = U(t) + iV(t)$ *with* $U'(t) = u(t)$, $V'(t) = v(t)$, *then*

$$\int_a^b w(t)dt = W(b) - W(a)$$

Proof.

By the fundamental theorem of calculus,

$$\int_a^b u(t)dt = U(b) - U(a) \quad \text{and} \quad \int_a^b v(t)dt = V(b) - V(a).$$

□

Example

Since

$$\frac{d}{dt}\frac{e^{it}}{i} = e^{it},$$

we have

$$\begin{aligned}\int_0^{\pi/4} e^{it}\,dt &= \left.\frac{e^{it}}{i}\right|_0^{\pi/4} = \left.-ie^{it}\right|_0^{\pi/4} \\ &= -i\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - 1\right) \\ &= \frac{\sqrt{2}}{2} + \left(1 - \frac{\sqrt{2}}{2}\right)i.\end{aligned}$$

Definition

A (parametrized) curve γ *is a set*

$$\gamma = \{z = z(t) = x(t) + y(t)i : t \in [a, b]\}, \qquad (*)$$

where $x(t)$ *and* $y(t)$ *are continuous real functions on* $[a, b]$. γ *is called a simple curve or a Jordan curve if it does not intersect itself, that is,* $z(t_1) \neq z(t_2)$ *unless* $t_1 = t_2$. γ *is called a simple closed curve if it does not intersect itself except for* $z(a) = z(b)$. *If* $x(t)$ *and* $y(t)$ *are continuously differentiable on* $[a, b]$*, then* γ *is called a smooth curve.* γ *is called a piecewise smooth curve if there are points*

$$a = a_0 < a_1 < ... < a_n = b,$$

such that $x(t)$ *and* $y(t)$ *are smooth in each interval* $[a_{k-1}, a_k]$*,* $k = 1, ..., n$.

Remark

(i) *The set defined by (*) is only a geometric object, which does not have a direction. But if we parametrize it by the parametrization* $z(t) = x(t) + y(t)i$*, then it is assigned a direction.*

(ii) *The length of a smooth curve* $\gamma = \{z = z(t) : t \in [a, b]\}$ *is*

$$\mathrm{length}(\gamma) = \int_a^b |z'(t)| dt.$$

If γ *is only piecewise smooth, its length is the sum of the lengths of its smooth parts.*

Definition

Given curve γ *defined in (*), we use* $-\gamma$ *to denote the same set of points of (*) but with reverse direction, say,*

$$-\gamma = \{z(a + b - t) : t \in [a, b]\}.$$

Definition

Two parametrization $z_1(t) : [a, b] \to \mathbb{C}$ *and* $z_2(t) : [c, d] \to \mathbb{C}$ *are called equivalent if there exists a continuously differentiable bijection* $s \mapsto t(s)$ *from* $[c, d]$ *to* $[a, b]$ *such that* $t'(s) > 0$ *and*

$$z_2(s) = z_1(t(s)).$$

Example

Here are some examples of curves.

(i) *The polygonal line defined by*

$$z(t) = \begin{cases} t + it, & t \in [0,1], \\ t + i, & t \in [1,2], \end{cases}$$

is a piecewise smooth curve.

(ii) *The unit circle with parametrization*

$$z(\theta) = e^{i\theta}, \quad \theta \in [0, 2\pi],$$

is a simple closed smooth curve.

(iii) *If γ be the unit circle defined in (ii). Then $-\gamma$ can be defined by the parametrization*

$$z(\theta) = e^{-i\theta}, \quad \theta \in [0, 2\pi],$$

(iv) *Given $m \in \mathbb{Z}\backslash\{0\}$, the curve defined by*

$$z(\theta) = e^{im\theta}, \quad \theta \in [0, 2\pi],$$

winds around the origin m times counterclockwise if $m > 0$. If $m < 0$, it winds around the origin m times clockwise.

Definition

Let γ be a smooth curve with parametrization $z(t)$, $t \in [a, b]$. If f is a continuous function on an open set Ω containing γ, we define

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

If γ is only piecewise smooth, which is smooth on intervals $[a_{k-1}, a_k]$, $k = 1, ..., n$, where $a = a_0 < a_1 < ... < a_n = b$, then

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} f(z(t))z'(t)dt.$$

Remark

The definition of integrals of functions along a curve γ is independent of the choice of the parametrization for γ. For

$$\gamma = \{z = z_1(t) : t \in [a, b]\}$$

and an equivalent parametrization $z_2 : [c, d] \to \mathbb{C}$ with

$$z_2(s) = z_1(t(s)), \quad t'(s) > 0,$$

we have

$$\begin{aligned}\int_a^b f(z_1(t))z_1'(t)dt &= \int_c^d f(z_1(t(s)))z_1'(t(s))t'(s)ds \\ &= \int_c^d f(z_2(s))z_2'(s)ds.\end{aligned}$$

Proposition

(i) *If* $c_1, c_2 \in \mathbb{C}$, *then*

$$\int_\gamma (c_1 f_1(z) + c_2 f_2(z))\, dz = c_1 \int_\gamma f_1(z)dz + c_2 \int_\gamma f_2(z)dz.$$

(ii)

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$$

(iii)

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma} |f(z)| \cdot \mathrm{length}(\gamma).$$

Proof.

Without loss of generality, we assume that γ is smooth. Part (i) follows the linearity of the Riemann integrals. For (ii), if

$$\gamma = \{z = z(t) : t \in [a, b]\},$$

we have

$$\begin{aligned}\int_{-\gamma} f(z)dz &= \int_a^b f(z(a+b-s))(z(a+b-s))' ds \\ &= -\int_a^b f(z(a+b-s))z'(a+b-s)ds \\ &= \int_b^a f(z(t))z'(t)dt = -\int_a^b f(z(t))z'(t)dt \\ &= -\int_\gamma f(z)dz.\end{aligned}$$

For (iii),

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|dt = \sup_{z\in\gamma} |f(z)| \cdot \mathrm{length}(\gamma).$$

□

Example

To evaluate

$$\int_\gamma \frac{dz}{z},$$

where $\gamma = \left\{ z = e^{i\theta} : \theta \in [0, \pi] \right\}$, *it holds*

$$\int_\gamma \frac{dz}{z} = \int_0^\pi \frac{1}{e^{i\theta}} i e^{i\theta} d\theta = i \int_0^\pi d\theta = \pi i.$$

Example

Let γ *be a smooth curve with parametrization* $z(t)$*,* $t \in [a, b]$*. Notice that*

$$\frac{d}{dt}(z(t))^2 = 2z(t)z'(t),$$

it holds

$$\int_\gamma z dz = \int_a^b z(t)z'(t)dt = \frac{1}{2}(z(t))^2\Big|_a^b = \frac{1}{2}\left((z(b))^2 - (z(a))^2\right).$$

Example

Let γ_1 *be the polygonal line starting from* 0 *to* i*, and then coming from* i *to* $1+i$*, then*

$$\begin{aligned}\int_{\gamma_1}\left(y-x-3x^2 i\right)dz &= \int_0^1 ti dt + \int_0^1 \left(1-t-3t^2 i\right)dt \\ &= \frac{i}{2}+\frac{1}{2}-i \\ &= \frac{1}{2}-\frac{i}{2}.\end{aligned}$$

Let γ_2 *be the line segment from* 0 *to* $1+i$*, then*

$$\begin{aligned}\int_{\gamma_2}\left(y-x-3x^2 i\right)dz &= \int_0^1 \left(t-t-3t^2 i\right)(1+i)dt \\ &= 1-i.\end{aligned}$$