MATH2230B
Complex Variables with Applications

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Example

In this example, we will check for what $c \in \mathbb{C}$, the function

$$f(z) := \begin{cases} z^c, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

is continuous at 0. *Here* z^c *is defined by using the following definition of the logarithm on* $\mathbb{C}\backslash\{0\}$:

$$\log z = \ln |z| + i \operatorname{Arg} z.$$

For $z \neq 0$,

$$z^c = e^{c \log z} = e^{c(\ln |z| + i\theta)}.$$

where $\theta = \operatorname{Arg} z$. *Suppose that* $c = c_1 + c_2 i$, $c_1, c_2 \in \mathbb{R}$, *then the above equality becomes*

$$z^c = e^{(c_1 \ln |z| - c_2\theta) + i(c_1\theta + c_2 \ln |z|)} = |z|^{c_1} e^{-c_2\theta} e^{i(c_1\theta + c_2 \ln |z|)}.$$

We divide it into three cases.

Example (continued)

(i) *For* $c_1 = 0$, *we have*

$$z^c = e^{-c_2\theta} e^{ic_2 \ln|z|}, \quad z \neq 0.$$

Taking the modulus of f,

$$|f(z)| = e^{-c_2\theta}.$$

If f *is continuous at* 0, *then* $\lim_{z\to 0} f(z) = 0$ *by the definition. Equivalently, it holds* $\lim_{z\to 0} |f(z)| = 0$. *Now, we first approach the origin along the ray with angle* 0. *We have*

$$|f(z)| = 1 \quad \textit{for all } z \textit{ with } \operatorname{Arg} z = 0.$$

Similarly, we can approach the origin along the ray with angle $\pi/2$ *and have*

$$|f(z)| = e^{-c_2\pi/2} \quad \textit{for all } z \textit{ with } \operatorname{Arg} z = \frac{\pi}{2}.$$

Therefore, if $c_2 \neq 0$, $|f|$ *is not continuous at* 0, *which leads a contradiction. As for the case* $c_1 = c_2 = 0$, *we have*

$$|f(z)| = 1 \quad \textit{for all } z \neq 0.$$

We conclude that f *is not continuous at* 0 *if* $c_1 = 0$.

Example (continued)

(ii) *For* $c_1 < 0$*, it holds*

$$|f(z)| = |z|^{c_1} e^{-c_2\theta} \quad \text{for all } z \neq 0.$$

In this case, for any θ *fixed, since* $c_1 < 0$*,*

$$\lim_{|z|\to 0} |f(z)| = \infty,$$

which implies that f *is not continuous at* 0*.*

(iii) *For* $c_1 > 0$*, it holds*

$$|f(z)| = |z|^{c_1} e^{-c_2\theta} \quad \text{for all } z \neq 0.$$

In this case, for any θ *fixed, since* $c_1 > 0$*,*

$$\lim_{|z|\to 0} |f(z)| = 0.$$

As a consequence, f *is continuous at* 0 *if* $c_1 > 0$*.*

In summary, f *is continuous at* 0 *if and only if* $\mathrm{Re}\, c > 0$*.*

Example

Let f be the principal square-root function defined by

$$f(z) = \begin{cases} |z|^{1/2} e^{i \operatorname{Arg} z/2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Then f is discontinuous on $S = \{z \in \mathbb{C} : \operatorname{Re} z < 0, \operatorname{Im} z = 0\}$. To see this, given a point $-R \in S$, $R > 0$, we can draw a circle centered at 0 with radius R. If we approach $-R$ along the circle from above, the limit equals to $\sqrt{R} e^{i\pi/2} = \sqrt{R} i$. On the other hand, if we approach $-R$ along the circle from below, the limit equals to $\sqrt{R} e^{-i\pi/2} = -\sqrt{R} i$. Consequently, f does not have a limit at $-R$, and thus is discontinuous there.

Definition

Let f be a function on an open set Ω. f is differentiable at $z_0 \in \Omega$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. And the limit, if it exists, is called the derivative of f at z_0 and denoted by $f'(z_0)$. The function f is said to be differentiable on Ω if it is differentiable at every point of Ω.

Example

Let $f(z) = 1/z$ on $\mathbb{C}\backslash\{0\}$. At each $z_0 \neq 0$, we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\frac{1}{z} - \frac{1}{z_0}}{z - z_0} = -\frac{1}{z_0 z}$$

Therefore,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = -\frac{1}{z_0^2}.$$

That is, f is differentiable at $z_0 \neq 0$, and $f'(z_0) = -\frac{1}{z_0^2}$.

Example

Let $f(z) = \overline{z}$ *on* $\mathbb{C}$. *For any* $z_0 \in \mathbb{C}$, *we have*

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{z} - \overline{z_0}}{z - z_0} = \frac{\overline{w}}{w},$$

where $w = z - z_0$. *Suppose that the limit* $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ *exists, then equivalently the limit* $\lim\limits_{w \to 0} \dfrac{\overline{w}}{w}$ *exists. It is easy to show that* $\dfrac{\overline{w}}{w}$ *does not have a limit at* 0. *Therefore,* f *is not differentiable at every* $z_0 \in \mathbb{C}$.

Example

Let $f(z) = c$ *for some* $c \in \mathbb{C}$*, then* f *is differentiable on* $\mathbb{C}$ *with*

$$f'(z) = 0.$$

Let $g(z) = z^n$ *for some* $n \in \mathbb{N}$*, then* g *is differentiable on* $\mathbb{C}$ *with*

$$g'(z) = nz^{n-1}.$$

Moreover, for a polynomial $P(z) = a_0 + a_1 z + ... + a_n z^n$*,* $a_0, a_1, ..., a_n \in \mathbb{C}$*,* P *is differentiable on* $\mathbb{C}$ *with*

$$P'(z) = a_1 + 2a_2 z + ... + na_n z^{n-1}.$$

Proposition

If f and g are differentiable functions on Ω, then

(i) *$f+g$ is differentiable on Ω, and $(f+g)' = f' + g'$.*

(ii) *fg is differentiable on Ω, and $(fg)' = f'g + fg'$.*

(iii) *If $g(z_0) \neq 0$ for $z_0 \in \Omega$, then f/g is differentiable at z_0, and*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Moreover, if $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \mathbb{C}$ are differentiable, then the composition $g \circ f$ is differentiable on Ω_1, and the chain rule holds

$$(g(f(z)))' = g'(f(z))\, f'(z).$$

Example

Let $f(z) = |z|^2$ *on* $\mathbb{C}$. *At each* $z_0 \in \mathbb{C}$, *we have*

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{|z|^2 - |z_0|^2}{z - z_0}.$$

By letting $w = z - z_0$,

$$|z|^2 = |w + z_0|^2 = (w + z_0)(\overline{w} + \overline{z_0}) = w\overline{w} + w\overline{z_0} + z_0\overline{w} + |z_0|^2.$$

Thus,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{w\overline{w} + w\overline{z_0} + z_0\overline{w}}{w} = \overline{w} + \overline{z_0} + z_0\,\frac{\overline{w}}{w}. \qquad (*)$$

If $z_0 = 0$, *(*) becomes*

$$\frac{f(z) - f(0)}{z - 0} = \overline{w},$$

which implies

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{w \to 0} \overline{w} = 0.$$

Hence, f *is differentiable at* 0 *with* $f'(0) = 0$.

Example (continued)

But if $z_0 \neq 0$*, the last term on the right-hand side of (*), i.e.,* $z_0 \dfrac{\overline{w}}{w}$*, has no limit as* $w \to 0$*. Therefore,* f *is not differentiable at every* $z_0 \neq 0$*.*

Remark

The last example illustrates the following facts.

(i) *A function can be differentiable at a point* z*, but nowhere else in any neighborhood of that point.*

(ii) *By writing a function* f *in the form* $f(z) = u(x,y) + iv(x,y)$*,* $z = x + yi$*, we may have* u *and* v *are both differentiable of all orders in variables* (x,y) *at a point* (x_0, y_0)*, but* f *is not differentiable at* $z_0 = x_0 + y_0 i$*.*

(iii) *The continuity of a function at a point does not imply the differentiability of the function there.*

Proposition

If f is differentiable at z_0, then f is continuous at z_0.

Proof.

$$\begin{aligned}\lim_{z\to z_0}(f(z)-f(z_0)) &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\lim_{z\to z_0}(z-z_0)\\ &= f'(z_0)\cdot 0\\ &= 0.\end{aligned}$$

□

Theorem

Let $f(z) = u(x, y) + iv(x, y)$*,* $z = x + yi$*, be defined on a neighborhood of* $z_0 = x_0 + y_0 i$*. If* f *is differentiable at* z_0*, then the partial derivatives of* u *and* v *exist and satisfy the Cauchy-Riemann equations*

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

at (x_0, y_0)*. Moreover,* $f'(z_0)$ *can be written as*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Proof.

Since $f'(z_0)$ exists, using the definition of $f'(z_0)$ and approaching $z_0 = x_0 + y_0 i$ by $(x_0 + h) + y_0 i$ with $h \in \mathbb{R}$,

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f((x_0 + h) + y_0 i) - f(x_0 + y_0 i)}{h} \\ &= \lim_{h \to 0} \left[\frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \right] \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0). \end{aligned}$$

On the other hand, we can also approach $z_0 = x_0 + y_0 i$ by $x_0 + (y_0 + h)i$ with $h \in \mathbb{R}$, which gives

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(x_0 + (y_0 + h)i) - f(x_0 + y_0 i)}{ih} \\ &= \lim_{h \to 0} \left[-i \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h} + \frac{v(x_0, y_0 + h) - v(x_0, y_0)}{h} \right] \\ &= v_y(x_0, y_0) - i u_y(x_0, y_0). \end{aligned}$$

Then we compete the proof by matching the real and imaginary parts of these two equalities. □

Example

We have shown that $f(z) = |z|^2$ *is differentiable only at* $z = 0$ *with* $f'(0) = 0$. *Notice that* $f(z) = u(x, y) + iv(x, y)$, $z = x + yi$, *with*

$$u(x, y) = x^2 + y^2 \quad \textit{and} \quad v(x, y) = 0.$$

It holds that u *and* v *satisfy the Cauchy-Riemann equations at* $(0, 0)$. *And we have*

$$f'(0) = 0 = u_x(0, 0) + iv_x(0, 0).$$

But f *cannot be differentiable at any* $z \neq 0$ *since* u *and* v *do not satisfy the Cauchy-Riemann equations there.*

Example

Let $f(z) = u(x,y) + iv(x,y)$*,* $z = x + yi$*, be defined by*

$$f(z) = \begin{cases} \overline{z}^2/z, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

then

$$u(x,y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \quad \text{and} \quad v(x,y) = \frac{-3x^2y + y^3}{x^2 + y^2}$$

if $(x,y) \neq (0,0)$*. Also,* $u(0,0) = v(0,0) = 0$*. Notice that*

$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

and

$$v_y(0,0) = \lim_{h \to 0} \frac{v(0,h) - v(0,0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

We have $u_x = v_y$ *at* $(0,0)$*.*

Example (continued)

Similarly, we have $u_y = -v_x = 0$ *at* $(0, 0)$*. That is, the Cauchy-Riemann equations are satisfied at* $z = 0$*. In contrast, for* $z \neq 0$,

$$\frac{f(z) - f(0)}{z - 0} = \left(\frac{\overline{z}}{z}\right)^2$$

does not have a limit as $z \to 0$*. To see this, if we approach* 0 *by* $z = \rho e^{i\theta_0}$ *for some fixed* $\theta_0 \in \mathbb{R}$ *and let* $\rho \to 0$*, we have*

$$\left(\frac{\overline{z}}{z}\right)^2 = e^{-4i\theta_0}.$$

We will get different limits as $\rho \to 0$ *with different* θ_0*'s.*

Theorem

Let $f(z) = u(x, y) + iv(x, y)$, $z = x + yi$, *be defined on an open set* Ω. *If* u *and* v *are continuously differentiable and satisfy the Cauchy-Riemann equations at* $z_0 = x_0 + y_0 i \in \Omega$, *then* f *is differentiable at* z_0 *with*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Proof.

By the continuous differentiability of u and v,

$$u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) = u_x(x_0, y_0)h_1 + u_y(x_0, y_0)h_2 + \varphi_1(h)|h|,$$
$$v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0) = v_x(x_0, y_0)h_1 + v_y(x_0, y_0)h_2 + \phi_2(h)|h|,$$

where $\varphi_1(h), \varphi_2(h) \to 0$ as $h \to 0$, $h = h_1 + h_2 i$. Then we have

$$\begin{aligned} & f(z_0 + h) - f(z_0) \\ = & \left(u_x(x_0, y_0) + iv_x(x_0, y_0)\right) h_1 + \left(u_y(x_0, y_0) + iv_y(x_0, y_0)\right) h_2 \\ & + \left(\varphi_1(h) + i\varphi_2(h)\right) |h|. \end{aligned}$$

Using the Cauchy-Riemann equations, the above equality becomes

$$\begin{aligned} & f(z_0 + h) - f(z_0) \\ = & \left(u_x(x_0, y_0) + iv_x(x_0, y_0)\right) h_1 + \left(-v_x(x_0, y_0) + iu_x(x_0, y_0)\right) h_2 \\ & + \left(\varphi_1(h) + i\varphi_2(h)\right) |h| \\ = & \left(u_x(x_0, y_0) + iv_x(x_0, y_0)\right) (h_1 + h_2 i) + \left(\varphi_1(h) + i\varphi_2(h)\right) |h|. \end{aligned}$$

By passing to the limit $h \to 0$, we complete the proof. □

Example

Recall the example for $f(z) = u(x,y) + iv(x,y)$*,* $z = x + yi$*, defined by*

$$f(z) = \begin{cases} \overline{z}^2/z & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Though u *and* v *satisfy the Cauchy-Riemann equations at* $(x,y) = (0,0)$*, the partial derivatives of* u *and* v *are not continuous at* $(0,0)$*. The assumptions of the last theorem do not holds.*

Example

Consider the function $f(z) = e^z = e^x(\cos y + i \sin y)$, *where* $z = x + yi$. *Then we have* $f(z) = u(x, y) + iv(x, y)$ *with*

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Notice that u *and* v *are both continuously differentiable and satisfy*

$$u_x = e^x \cos y = v_y \quad \text{and} \quad u_y = -e^x \sin y = -v_x.$$

for all $(x, y) \in \mathbb{R}^2$. *Therefore,* f *is differentiable on* $\mathbb{C}$ *with*

$$f' = u_x + iv_x = e^x \cos y + ie^x \sin y.$$

Note that $f'(z) = f(z)$ *for all* $z \in \mathbb{C}$.

Example

Let $f(z) = x^3 + i(1-y)^3$, $z = x + yi$. *Then* $f(z) = u(x,y) + iv(x,y)$ *with*

$$u(x,y) = x^3 \quad \text{and} \quad v(x,y) = (1-y)^3.$$

First, notice that u *and* v *are continuously differentiable on* $\mathbb{R}^2$. *As for the Cauchy-Riemann equations,*

$$\begin{array}{ll} u_x = 3x^2, & u_y = 0, \\ v_x = 0, & v_y = -3(1-y)^2. \end{array}$$

Then we always have $u_y = -v_x$. *But* $u_x = v_y$ *only if* $(x,y) = (0,1)$. *Therefore,* f *is differentiable only at* $z = i$ *with*

$$f'(i) = u_x(0,1) + iv_x(0,1) = 0.$$

Example

Let $f(z) = \sin x \cosh y + i \cos x \sinh y$, $z = x + yi \in \mathbb{C}$. *Then* $f = u + iv$ *with*

$$u(x, y) = \sin x \cosh y \quad \textit{and} \quad v(x, y) = \cos x \sinh y.$$

Since u *and* v *are continuously differentiable and satisfy*

$$u_x = \cos x \cosh y = v_y \quad \textit{and} \quad u_y = \sin x \sinh y = -v_x$$

everywhere, we conclude that f *is differentiable on* $\mathbb{C}$ *with*

$$f'(z) = u_x + iv_x = \cos x \cosh y - i \sin x \sinh y.$$