MATH2230B
Complex Variables with Applications

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Definition

Given a function f defined on an open set Ω, and z_0 an accumulation point of $\Omega\backslash\{z_0\}$, we call that f has a limit w_0 at z_0 if for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \textit{for all} \quad z \in \Omega, \quad 0 < |z - z_0| < \delta.$$

And we write it as

$$\lim_{z \to z_0} f(z) = w_0.$$

Example

Check that if the function

$$f(z) := z/\overline{z}, \quad z \neq 0,$$

has a limit at 0*. Notice that, for* $z = x + yi \neq 0$,

$$\operatorname{Re} f = \frac{x^2 - y^2}{x^2 + y^2}.$$

If we approach the origin along the real axis, for all $x \in \mathbb{R}$*, we have*

$$\operatorname{Re} f(x) = 1.$$

And if we approach the origin along the imaginary axis, for all $y \in \mathbb{R}$*, we have*

$$\operatorname{Re} f(yi) = -1.$$

Therefore, f *does not have a limit at* 0*.*

Example

Let $f(z) = 1/\log z$ *be defined by using the principal branch of the logarithm on* $\Omega = \{z \in \mathbb{C} : |z| > 0, -\pi < \mathrm{Arg}\, z < \pi\}$. *To show that*

$$\lim_{z \to 0} f(z) = 0,$$

we recall that for $z \in \Omega$,

$$f(z) = \frac{1}{\log z} = \frac{1}{\ln|z| + i\mathrm{Arg}\, z}.$$

Hence,

$$|f(z)| \leq \frac{1}{\sqrt{(\ln|z|)^2 + (\mathrm{Arg}\, z)^2}} \leq -\frac{1}{\ln|z|}$$

for all $z \in \Omega$ *with* $|z| < 1$. *Therefore,*

$$|f(z)| < \varepsilon \quad \textit{provided} \quad |z| < \min\left\{1, e^{-1/\varepsilon}\right\}.$$

Definition

Given a function f defined on an open set $\Omega \subset \mathbb{C}$, we call that f is continuous at $z_0 \in \Omega$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If f is continuous at every point $z \in \Omega$, we call that f is continuous on Ω.

Proposition

If f and g are functions on an open set $\Omega \subset \mathbb{C}$ and continuous at $z_0 \in \Omega$, then $f + g$ and fg are both continuous at z_0. Moreover, if $g(z_0) \neq 0$, then f/g is also continuous at z_0.

Proposition

For $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \mathbb{C}$, where Ω_1 and Ω_2 are open sets in $\mathbb{C}$, suppose that f is continuous at $z_0 \in \Omega_1$ and that g is continuous at $f(z_0)$, then the composition $g \circ f$ is continuous at z_0.

Example

$\mathrm{Re}\, z$, $\mathrm{Im}\, z$, $|z|$ and $\overline{z}$ are all continuous functions. If $f : \Omega \to \mathbb{C}$ is continuous on an open set $\Omega \subset \mathbb{C}$, then $|f(z)|$ is also continuous on Ω.

Example

Define the function

$$f(z) := \begin{cases} z/\overline{z}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Noticing that f does not have a limit at 0, f is not continuous at 0.