MATH2230B
Complex Variables with Applications

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Example (Lines)

Given two complex numbers z_1 and z_2, they determine a straight line L such that L passes across z_1 and z_2. For all points on L, denoted by z, the direction from z_1 to z_2 and the direction from z_1 to z are either the same or different by π. Therefore, by polar coordinates, if $z_2 - z_1 = \rho e^{i\theta}$, then it must hold

$$z - z_1 = re^{i\theta} \quad \text{or} \quad z - z_1 = re^{i(\theta+\pi)}.$$

Here ρ and r are moduli of $z_2 - z_1$ and $z - z_1$, respectively. Therefore, we have

$$\text{either} \quad \frac{z - z_1}{z_2 - z_1} = \frac{r}{\rho} \quad \text{or} \quad \frac{z - z_1}{z_2 - z_1} = -\frac{r}{\rho}.$$

In either case, $\dfrac{z - z_1}{z_2 - z_1}$ is real, provided that z lies on the line L. The converse is also true. So in the complex theory, line L determined by z_1 and z_2 can be represented by

$$L = \left\{ z \in \mathbb{C} : \mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \right\}.$$

Example

Find all points which satisfy

$$\operatorname{Im}\left(\frac{z+1-3i}{4-i}\right)=0.$$

The condition given in this example is quite similar to the formula of lines. It is a particular case when we have

$$-z_1 = 1-3i \quad \text{and} \quad z_2 - z_1 = 4-i.$$

That is, $z_1 = -1+3i$ *and* $z_2 = 3+2i$*. By the discussion in the last example, the points in this example represent a line passing across* $-1+3i$ *and* $3+2i$*.*

Example (Sides of a line)

Given different z_1 and z_2 in $\mathbb{C}$, we can determine a line L. There are two directions if a line is given. One direction is from z_1 to z_2, while another direction is from z_2 to z_1. The concept of side is related to the direction that we are using. If we fix a direction by starting from z_1 to z_2, then all points on the left form the left-hand side of the line L, while all points on the right form the right-hand side of the line L. Pay attention that the left-hand side and the right-hand side depend on the direction that we are using. Suppose that the direction is given by starting from z_1 to z_2. Then, for an arbitrary point z on the left-hand side, we can rotate $z_2 - z_1$ counterclockwise by an angle $\theta_0 \in (0, \pi)$ to the direction given by $z - z_1$. In other words,

$$z - z_1 = \lambda_0 (z_2 - z_1) e^{i\theta_0},$$

for some $\lambda_0 > 0$ and $\theta_0 \in (0, \pi)$.

Example (Sides of a line, continued)

From the above equality, we have

$$\mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = \lambda_0 \sin\theta_0 > 0.$$

Similarly, if z is on the right-hand side of L with the direction given by pointing from z_1 to z_2, then it holds

$$\mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = \lambda_0 \sin\theta_0 < 0.$$

The above arguments and the formula of lines implies that given z_1 and z_2, all points satisfy the formula of lines must lie on the line across z_1 and z_2. If

$$\mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) > 0,$$

then z lies on the left-hand side of L with the direction from z_1 to z_2. If

$$\mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) < 0,$$

then z lies on the right-hand side of L.

Example

Question: Find all points satisfying

$$\mathrm{Im}\left(\frac{z+1-3i}{4-i}\right) > 0.$$

Recall that points satisfy

$$\mathrm{Im}\left(\frac{z+1-3i}{4-i}\right) = 0$$

lie on the line L *across* $z_1 = -1+3i$, $z_2 = 3+2i$. *Therefore, those points* z *satisfying*

$$\mathrm{Im}\left(\frac{z+1-3i}{4-i}\right) > 0.$$

must be on the left-hand side of L *with the direction from* z_1 *to* z_2.

Example (Reflection in the real axis)

In complex theory, given a complex number $z = x + yi$*, we have an operator to find its symmetric point with respect to the* x*-axis. In fact, the symmetric point of* (x, y) *with respect to the* x*-axis is* $(x, -y)$*. This symmetric point corresponds to the number* $x - yi$*. In the future, we denote by* $\overline{z} = x - yi$ *the symmetric point of* z *with respect to the* x*-axis.*

Definition (Complex conjugates)

For $z = x + yi \in \mathbb{C}$*, the symmetric point of* z *with respect to the real axis, i.e.,*

$$\overline{z} = x - yi,$$

is called the conjugate of z*.*

Property

(i) $\overline{\overline{z}} = z$ *and* $|\overline{z}| = |z|$.

(ii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$, *and* $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$. *If* $z_2 \neq 0$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

(iii) $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ *and* $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$.

(iv) $z\overline{z} = |z|^2$.

Example (Computation of roots)

Given $z = \rho e^{i\theta}$*, we can easily calculate* $z^n = \rho^n e^{in\theta}$*. Conversely, if we are given* $a = \rho_0 e^{i\theta_0} \neq 0$*, we can also find* z *such that* $z^n = a$*,* $n \in \mathbb{N}$*. Indeed, suppose that* $z = \rho e^{i\theta}$*, then* $z^n = a$ *can be equivalently written as*

$$\rho^n e^{in\theta} = \rho_0 e^{i\theta_0}.$$

It then follows

$$\rho = \sqrt[n]{\rho_0} \quad \text{and} \quad e^{i\left(n\theta - \theta_0\right)} = 1.$$

ρ *is uniquely determined. But cosine and sine are periodic function, the second equality above can only imply*

$$n\theta - \theta_0 = 2k\pi,$$

for some $k \in \mathbb{Z}$*. Therefore,* θ *is not uniquely determined. All* z *with* $\rho = \rho_0^{1/n}$ *and* θ *given by*

$$\frac{\theta_0}{n} + \frac{2k\pi}{n}, \quad k \in \mathbb{Z},$$

will satisfy the equation $z^n = a$*. Such* z *is called an* n*-th root of* a*. Notice that we can only have* n *different roots for a given non-zero complex number* a*.*

Definition

For $z \in \mathbb{C}$, $n \in \mathbb{N}$, *we denote* $z^{1/n}$ *the set of* n*-th roots of* z. *If* $z = \rho e^{i\theta} \neq 0$,

$$z^{1/n} = \left\{ \sqrt[n]{\rho} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} : k = 0, 1, ..., n-1 \right\}.$$

In particular, if $z = \rho e^{i\theta} \neq 0$ *with* $\theta \in (-\pi, \pi]$, *i.e.,* $\theta = \mathrm{Arg}\, z$, *then*

$$\sqrt[n]{\rho} e^{i\theta/n} = \sqrt[n]{\rho} e^{i\mathrm{Arg} z/n}$$

is called the principal n*-th root of* z.

Remark

If $z = 0$, *all the* n*-th roots are* 0.

Example

To find all of the fourth roots of -16*, we have*

$$-16 = 16e^{i\pi}.$$

Therefore,

$$(-16)^{1/4} = \left\{2e^{i\pi/4}, 2e^{i3\pi/4}, 2e^{i5\pi/4}, 2e^{i7\pi/4}\right\}.$$

Example

To find all of the n-th roots of 1*, we notice that*

$$1 = 1e^{i\cdot 0}.$$

Therefore,

$$1^{1/n} = \left\{e^{i(2k\pi/n)} : k = 0, 1, ..., n-1\right\}.$$

Definition

Given a set $S \subset \mathbb{C}$, a point $z_0 \in \mathbb{C}$ is called an interior point of S if there is $r_0 > 0$ such that

$$B_{r_0}(z_0) = \{z \in \mathbb{C} : |z - z_0| < r_0\} \subset S.$$

A point $z_0 \in \mathbb{C}$ is called an exterior point of S if there is $r_1 > 0$ such that

$$B_{r_1}(z_0) = \{z \in \mathbb{C} : |z - z_0| < r_1\} \subset \mathbb{C} \backslash S.$$

A point z_0 is a boundary point of S if it is neither an interior point nor an exterior point of S. A point z_0 is an accumulation point or a limit point if for any $r > 0$,

$$B_r(z_0) \cap S \neq \phi.$$

Definition

For a set $S \subset \mathbb{C}$, the interior of S consists of all its interior points. We said that S is open if every point in S is an interior point.

Definition

A set S is closed if the complement $\mathbb{C}\backslash S$ is open. The closure of S is the closed set consists of all points of S and its boundary.

Remark

(i) ϕ *and* $\mathbb{C}$ *are both open and closed.*

(ii) *A set can be neither open nor closed. For example, the set* $S = \{z \in \mathbb{C} : 1 < |z| \leq 2\}$ *is neither open nor closed.*

(iii) *For the set S in (ii), the interior of S is* $\{z \in \mathbb{C} : 1 < |z| < 2\}$, *and the closure of S is* $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$.

Definition

A set S is bounded if there is $M > 0$ such that $|z| \leq M$ for all $z \in S$.

Definition

If S is a bounded set, we can define the diameter of S by

$$\operatorname{diam}(S) = \sup_{z_1, z_2 \in S} |z_1 - z_2|.$$

Definition

A set S is connected if it cannot be partitioned into two part $S = S_1 \cup S_2$ for nonempty S_1, S_2 such that

$$S_1 \subset U \quad \text{and} \quad S_2 \subset V$$

where U and V are disjoint open sets.

Definition

Given a set S, we call a family of open sets $\{O_\alpha\}$ is an open covering of S if

$$S \subset \bigcup_{\alpha} O_\alpha.$$

Definition

A set S is called compact if every open covering of S has a finite subcovering.

Proposition

A set S is compact if and only if S is closed and bounded.

Proposition

If we have a sequence of non-empty compact sets $S_1 \supset S_2 \supset \ldots \supset S_n \supset \ldots$ with $\mathrm{diam}(S_n) \to 0$ as $n \to \infty$, then there is a unique $z_0 \in \mathbb{C}$ such that $z_0 \in S_n$ for all $n \in \mathbb{N}$.

Definition

Let S_1 and S_2 be subsets of $\mathbb{C}$. A function f is defined on S_1 if for each $z \in S_1$, there is a unique complex number $f(z) \in S_2$. We write it as

$$f : S_1 \longrightarrow S_2.$$

The set S_1 is called the domain of f.

Remark

A complex function f on S can be represented as

$$f = f_1 + f_2 i,$$

where f_1 and f_2 are two real-valued functions defined on S.

Here are some examples of functions.

Example

$f(z) = z^2$ defined on $\mathbb{C}$. If $z = x + yi$, then

$$f(z) = \left(x^2 - y^2\right) + 2xyi.$$

$g(z) = |z|^2$ defined on $\mathbb{C}$. We have, for $z = x + yi$,

$$g(z) = x^2 + y^2.$$

Example

For $n \in N$, given $n + 1$ complex numbers $a_0, a_1, ..., a_n$, then the function

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

is called a polynomial of degree n. P can be defined on the whole $\mathbb{C}$.

Example

Let $P(z)$ and $Q(z)$ be two polynomials. The quotient $P(z)/Q(z)$ is called a rational function and is defined at each point z with $Q(z) \neq 0$. For example, the function

$$R(z) = \frac{z^2+3}{z^3+z^2+5z+5} = \frac{z^2+3}{(z+1)(z^2+5)}$$

is defined on $\mathbb{C} \setminus \left\{-1, \sqrt{5}i, -\sqrt{5}i\right\}$.

Example

We know that 0 is the only square root for 0. But for a complex number $z \neq 0$, the square roots of a complex number z are

$$z^{1/2} = \left\{ \sqrt{|z|} e^{i \operatorname{Arg} z/2}, -\sqrt{|z|} e^{i \operatorname{Arg} z/2} \right\},$$

which consists of two values. So $z^{1/2}$ is not a function. But if we particularly choose one of them, say, we define

$$f(z) = \begin{cases} \sqrt{|z|} e^{i \operatorname{Arg} z/2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Then f is a function on $\mathbb{C}$. More generally, given any $\theta_0 \in \mathbb{R}$, we can define a function

$$g(z) = \begin{cases} \sqrt{|z|} e^{i\theta/2} & \text{if } z = |z| e^{i\theta} \neq 0, \quad \theta \in (\theta_0, \theta_0 + 2\pi], \\ 0 & \text{if } z = 0, \end{cases}$$

which also corresponds to a square root of z.

Example

(i) *For* $z_0 \in \mathbb{C}$, $f_1(z) := z + z_0$, *which is a translation function.*
(ii) *For* $\theta_0 \in \mathbb{R}$, $f_2(z) := e^{i\theta_0} z$, *which is a rotation function.*
(iii) *For* $r_0 \in \mathbb{R}$, $f_3(z) := r_0 z$, *which is a scaling function.*
(iv) $f_4(z) := \overline{z}$, *which corresponds to the reflection with respect to the real axis.*

All of these functions are defined on $\mathbb{C}$.

Example

Given $c \in \mathbb{C}$, *the function* e^{cz} *is defined on* $\mathbb{C}$.

Example

We define the sine and the cosine for complex numbers by

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

Also, the hyperbolic sine and the hyperbolic cosine are defined by

$$\cosh z := \frac{e^{z} + e^{-z}}{2} \quad \text{and} \quad \sinh z := \frac{e^{z} - e^{-z}}{2}.$$

All of these functions are defined on $\mathbb{C}$.

Example (Complex logarithm)

The motivation of the definition of the logarithm is to find the inverse of the exponential function. That is, we want to solve the equation

$$e^z = w$$

for given $w \in \mathbb{C} \backslash \{0\}$. *Suppose that* $w = \rho e^{i\theta}$, $\rho = |w|$, $\theta = \operatorname{Arg} w$, *and* $z = x + yi$, *then the above equation becomes*

$$e^{x+yi} = \rho e^{i\theta}.$$

We have,

$$e^x = \rho \quad \text{and} \quad e^{iy} = e^{i\theta},$$

which gives

$$x = \ln \rho \quad \text{and} \quad y = \theta + 2k\pi, \quad k \in \mathbb{Z}.$$

Here ln *denotes the logarithm for the real numbers.*

Example (Complex logarithm, continued)

There is a multi-value problem. If we fix $\alpha_0 \in \mathbb{R}$, then for each $\theta \in (-\pi, \pi]$, we can determine a unique $k \in \mathbb{Z}$ such that

$$\theta + 2k\pi \in (\alpha_0, \alpha_0 + 2\pi].$$

Then we can define

$$\log z := \ln \rho + i\,(\theta + 2k\pi) \quad \textit{such that} \quad \theta + 2k\pi \in (\alpha_0, \alpha_0 + 2\pi],$$

which is a function on $\mathbb{C}\backslash\{0\}$.

Definition (Principal branch of the logarithm)

A branch of the logarithm is a continuous function f defined on an open subset U of $\mathbb{C}\backslash\{0\}$ such that

$$e^{f(z)} = z$$

for all $z \in U$. The principal branch of the logarithm is defined by

$$\log z := \ln |z| + i\,\mathrm{Arg}\, z$$

on $\{z : \mathbb{C} : |z| > 0, -\pi < \mathrm{Arg}\, z < \pi\}$.

Example

Given a branch of the logarithm defined on U and a complex number c, we can define the power function z^c to be

$$z^c = e^{c \log z}.$$

If the principal branch of the logarithm is used, the above definition is called the principal branch of the power function z^c.

Remark

In general, given $c \in \mathbb{C}$, z^c might not be defined at 0.

Example

By using the principal branch of the power function z^i,

$$i^i = e^{i \log i} = e^{i\left(\ln 1 + \frac{\pi i}{2}\right)} = e^{-\pi/2}.$$

Definition

Given a function f defined on an open set Ω, and z_0 an accumulation point of $\Omega \backslash \{z_0\}$, we call that f has a limit w_0 at z_0 if for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \textit{for all} \quad z \in \Omega, \quad 0 < |z - z_0| < \delta.$$

And we write it as

$$\lim_{z \to z_0} f(z) = w_0.$$

Proposition

(i) *If* $\lim_{z\to z_0} f(z) = w_1$ *and* $\lim_{z\to z_0} f(z) = w_2$*, then* $w_1 = w_2$*.*

(ii) *If* $f(z) = u(z) + iv(z)$*, where* u *and* v *are real-valued functions, then* $\lim_{z\to z_0} f(z) = u_0 + v_0 i$ *if and only if*

$$\lim_{z\to z_0} u(z) = u_0 \quad \textit{and} \quad \lim_{z\to z_0} v(z) = v_0.$$

(iii) *If* $\lim_{z\to z_0} f(z) = w_1$ *and* $\lim_{z\to z_0} g(z) = w_2$*, then*

$$\lim_{z\to z_0} (f(z) + g(z)) = w_1 + w_2$$

and

$$\lim_{z\to z_0} (f(z)g(z)) = w_1 w_2.$$

If, in addition, $w_2 \neq 0$*, then*

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_1}{w_2}.$$

Example

To show that if $f(z) = i\bar{z}/2$, *then*

$$\lim_{z \to 1} f(z) = \frac{i}{2}.$$

Notice that

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}.$$

We have

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \textit{provided} \quad |z-1| < 2\varepsilon.$$

Example

For a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ *with* $a_0, \ldots, a_n \in \mathbb{C}$, $n \in \mathbb{N}$, *we have the limit*

$$\lim_{z \to z_0} P(z) = a_0 + a_1 z_0 + a_2 z_0^2 + \ldots + a_n z_0^n.$$