MATH2230B
Complex Variables with Applications

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Proposition (Triangle inequality, generalization)

For $z_1, \ldots, z_n \in \mathbb{C}$*,*

$$|z_1 + \ldots + z_n| \leq |z_1| + \ldots + |z_n|.$$

Proof.

By mathematical induction. □

Example

We can use the triangle inequality to estimate $3 + z + z^2$ *for all* z *with* $|z| = 2$ *as follows. By the triangle inequality,*

$$|3 + z + z^2| \leq 3 + |z| + |z^2|.$$

Since $|z^2| = |z|^2$ *and* $|z| = 2$*, the above estimate is reduced to*

$$|3 + z + z^2| \leq 3 + |z| + |z^2| = 3 + |z| + |z|^2 = 9.$$

Multiplication

For two complex numbers $z_1 = \rho_1 e^{i\theta_1}$ *and* $z_2 = \rho_2 e^{i\theta_2}$*, we have*

$$z_1 z_2 = \rho_1 \rho_2 e^{i(\theta_1 + \theta_2)}.$$

That is,

$$|z_1 z_2| = \rho_1 \rho_2$$

and

$$\arg(z_1 z_2) = \{\theta_1 + \theta_2 + 2k\pi : k \in \mathbb{Z}\}.$$

Remark

When a complex number $z_1 = \rho_1 e^{i\theta_1}$ *is multiplied by another complex number* $z_2 = \rho_2 e^{i\theta_2}$*, we have the modulus of the product* $|z_1 z_2| = \rho_2 |z_1|$*. That is, it corresponds to stretch or compress the vector* z_1*. Since* $\theta_1 + \theta_2$ *is an argument of* $z_1 z_2$*, the direction of* $z_1 z_2$ *can be obtained by rotating the direction of* z_1 *counterclockwise by* θ_2 *if* $\theta_2 > 0$*, or clockwise by* $-\theta_2$ *if* $\theta_2 < 0$*.*

Remark

(i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ *in the sense of set addition. But in general, the equality*

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

is false.

(ii) *For complex number* $z = \rho e^{i\theta}$, $\rho > 0$, $z^{-1} = \rho^{-1} e^{-i\theta}$.

(iii) *For* $z \neq 0$, $\arg(z^{-1}) = -\arg z$.

(iv) *For* $z_1, z_2 \in \mathbb{C}$, $z_2 \neq 0$, $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

(v) *For complex number* $z = \rho e^{i\theta}$, $\rho > 0$, $z^n = \rho^n e^{in\theta}$ *for all* $n \in \mathbb{Z}$.

(vi) *(de Moivre's formula) By using (v) with* $\rho = 1$, *for* $n \in \mathbb{Z}$, *we have*

$$\left(e^{i\theta}\right)^n = e^{in\theta}.$$

That is,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Example

If $z_1 = -1$ and $z_2 = i$, then

$$\operatorname{Arg} z_1 = \pi \quad \textit{and} \quad \operatorname{Arg} z_2 = \frac{\pi}{2}.$$

However,

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(-i) = -\frac{\pi}{2} \neq \frac{3\pi}{2} = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

Example

In order to find the principal argument of $z = \dfrac{i}{-1-i}$, we start by writing

$$\arg z = \arg i - \arg(-1-i).$$

Since

$$\operatorname{Arg} i = \frac{\pi}{2} \quad \textit{and} \quad \operatorname{Arg}(-1-i) = -\frac{3\pi}{4},$$

we have that $\dfrac{5\pi}{4} \in \arg z$. Therefore,

$$\operatorname{Arg} z = -\frac{3\pi}{4}.$$

Example

By de Moivre's formula with $n = 2$, we have

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta.$$

That is,

$$\left(\cos^2\theta - \sin^2\theta\right) + i\left(2\sin\theta\cos\theta\right) = \cos 2\theta + i\sin 2\theta.$$

Therefore,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \textit{and} \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Example (Circles and discs)

A circle with center z_0 and radius r_0 is given by

$$\{z \in \mathbb{C} : |z - z_0| = r_0\}$$

The interior part of the circle is the set

$$\{z \in \mathbb{C} : |z - z_0| < r_0\}.$$

The exterior part of the circle is the set

$$\{z \in \mathbb{C} : |z - z_0| > r_0\}.$$

Example (Ellipses)

An ellipse with foci z_1 and z_2 is given by $\{z \in \mathbb{C} : |z - z_1| + |z - z_2| = d\}$. *Here d is the length of the long axis.*

Example (Another representation for circles)

A circle can be uniquely determined by given three points which are not on the same line. Suppose that C is the circle passing across z_1, z_2 and z_3. For another point z on C, without loss of generality, we assume that z_1, z_2, z_3 and z are clockwise distributed. Other cases can be similarly considered. Then by fundamental geometry, it holds

$$\angle z_1 z_3 z_2 = \angle z_1 z z_2.$$

The reason is that these two angles correspond to the same arc on the circle C. Notice that we can rotate the vector $z_3 - z_2$ counterclockwise by the angle $\angle z_1 z_3 z_2$, the resulted vector must have the same direction as $z_3 - z_1$. Therefore, we have

$$z_3 - z_1 = \lambda_1 (z_3 - z_2) e^{i \angle z_1 z_3 z_2}$$

for some $\lambda_1 > 0$. Similarly, we have

$$z - z_1 = \lambda_2 (z - z_2) e^{i \angle z_1 z z_2}$$

for some $\lambda_2 > 0$.

Example (Another representation for circles, continued)

Here λ_1 *and* λ_2 *are positive real numbers. Since* $\angle z_1 z_3 z_2 = \angle z_1 z z_2$, *the last two equalities yield*

$$\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \frac{\lambda_2}{\lambda_1}.$$

This furthermore implies

$$\mathrm{Im}\left[\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right)\right] = 0.$$

One can apply similar arguments above for the other possible positions of z *on* C. *The last equality always holds once* z *is on* C. *Therefore, we conclude that*

$$C = \left\{ z \in \mathbb{C} : \mathrm{Im}\left[\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right)\right] = 0 \right\}.$$

Example

Question: Find all points which satisfy

$$\mathrm{Im}\left(\frac{1}{z}\right) = 1.$$

Notice that

$$\mathrm{Im}\left(\frac{1}{z}\right) = 1 = \mathrm{Im}\,(i)\,.$$

Therefore,

$$0 = \mathrm{Im}\left(\frac{1}{z} - i\right) = \mathrm{Im}\left(\frac{1 - iz}{z}\right) = \mathrm{Im}\left(\frac{z + i}{z} \cdot (-i)\right).$$

Compare with the formula for circles, we have in this example

$$-z_1 = i, \quad z_2 = 0, \quad \textit{and} \quad \frac{z_3 - z_2}{z_3 - z_1} = -i.$$

Equivalently, it holds $z_1 = -i$, $z_2 = 0$, $z_3 = \frac{1}{2} - \frac{i}{2}$. *It represents a circle passing across these three points. Analytically all points in this example satisfy*

$$\left|z + \frac{i}{2}\right| = \frac{1}{2}.$$