MATH2230B
Complex Variables with Applications

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Some specific sets:

$$\mathbb{N} = \text{the set of all natural numbers} = \{1, 2, 3, ...\};$$

$$\mathbb{Z} = \text{the set of all integers} = \{..., -2, -1, 0, 1, 2, ...\};$$

$$\mathbb{Q} = \text{the set of all rational numbers} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\};$$

$$\mathbb{R} = \text{the set of all real numbers.}$$

Definition

By introducing the pure imaginary number i satisfying $i^2 = -1$, the set of complex numbers $\mathbb{C}$ is defined by

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}.$$

Definition

For a complex number $z = x + yi$, $x, y \in \mathbb{R}$, x *and* y *are the real and imaginary parts of* z. *We denote*

$$\mathrm{Re}\, z = x \quad \textit{and} \quad \mathrm{Im}\, z = y.$$

If $\mathrm{Im}\, z = 0$, *then* z *is a real number. If* $\mathrm{Re}\, z = 0$, z *is called a pure imaginary number. Two complex numbers* z_1 *and* z_2 *are equal if*

$$\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2 \quad \textit{and} \quad \mathrm{Im}\, z_1 = \mathrm{Im}\, z_2.$$

Remark

For $a, b \in \mathbb{R}$, *one of the following three relations holds: (i)* $a < b$; *(ii)* $a = b$; *(iii)* $a > b$. *But for complex numbers* z_1 *and* z_2, *we do not have* $z_1 > z_2$ *or* $z_1 < z_2$.

Definition (Addition)

For $z_1 = x_1 + y_1 i$ *and* $z_2 = x_2 + y_2 i$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, *we define the sum* $z_1 + z_2$ *to be*

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i.$$

Property

(i) *(Commutative law)* $z_1 + z_2 = z_2 + z_1$ *for all* $z_1, z_2 \in \mathbb{C}$.

(ii) *(Associative law)* $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$.

(iii) *(Summation identity) There is* $0 \in \mathbb{C}$ *such that* $z + 0 = z$ *for all* $z \in \mathbb{C}$.

(iv) *(Summation inverse) For all* $z \in \mathbb{C}$, *there is* $-z \in \mathbb{C}$ *such that* $z + (-z) = 0$.

Remark

(i) $0 = 0 + 0i$.

(ii) *For* $z = x + yi$, $x, y \in \mathbb{R}$, $-z = (-x) + (-y)i$.

Definition (Subtraction)

For $z_1, z_2 \in \mathbb{C}$, *we define the subtraction* $z_1 - z_2$ *to be*

$$z_1 - z_2 = z_1 + (-z_2).$$

Formal Calculation

Assuming that the commutative law, associative law and the distributive law hold for complex numbers, for $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned}(x_1 + y_1 i)(x_2 + y_2 i) &= x_1x_2 + x_1y_2 i + x_2y_1 i + y_1y_2 i^2 \\ &= x_1x_2 + x_1y_2 i + x_2y_1 i - y_1y_2 \\ &= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i.\end{aligned}$$

Definition (Multiplication)

For $z_1 = x_1 + y_1 i$ *and* $z_2 = x_2 + y_2 i$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, *we define the product* $z_1 z_2$ *to be*

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i.$$

Property

(i) *(Commutative law)* $z_1 z_2 = z_2 z_1$ *for all* $z_1, z_2 \in \mathbb{C}$.

(ii) *(Associative law)* $z_1(z_2 z_3) = (z_1 z_2) z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$.

(iii) *(Distributive law)* $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$.

(iv) *(Multiplication identity) There is* $1 \in \mathbb{C}$ *such that* $z \cdot 1 = z$ *for all* $z \in \mathbb{C}$.

(v) *(Multiplication inverse) For all* $z \in \mathbb{C} \backslash \{0\}$, *there is* $z^{-1} \in \mathbb{C}$ *such that* $z z^{-1} = 1$.

Remark

(i) *If $z_1, z_2 \in \mathbb{C}$, $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$, or possibly $z_1 = z_2 = 0$.*

(ii) $1 = 1 + 0i$.

(iii) *For $z = x + yi$, $x, y \in \mathbb{R}$, $z^{-1} = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2} i$.*

(iv) *Sometimes, we denote z^{-1} by $\frac{1}{z}$.*

(v) *For $z \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, z^n is defined inductively by*

$$\begin{cases} z^k = z^{k-1} z & \text{for } k \in \mathbb{N}, \\ z^0 = 1. \end{cases}$$

Remark

(vi) *(Binomial formula) For* $z_1, z_2 \in \mathbb{C}\backslash\{0\}$, $n \in \mathbb{N}$,

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k},$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Definition (Division)

For $z_1, z_2 \in \mathbb{C}$, $z_2 \neq 0$, *we define the division by*

$$\frac{z_1}{z_2} = z_1 z_2^{-1}.$$

Remark

For $z_1, \ldots, z_4 \in \mathbb{C}$, $z_3 \neq 0$, $z_4 \neq 0$,

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4}.$$

Example

$$\frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{(2+3i)(2-3i)} = \frac{5+14i}{13} = \frac{5}{13} + \frac{14}{13}i.$$

Definition (Euler's formula)

For $y \in \mathbb{R}$,

$$e^{yi} = \cos y + i \sin y.$$

Formal Calculation

Recall that the exponential function for real numbers admits a Taylor expansion. For $x \in \mathbb{R}$,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

If the above expansion holds for complex numbers, particularly for pure imaginary numbers, we have

$$e^{yi} = \sum_{n=0}^{\infty} \frac{y^n i^n}{n!}.$$

Formal Calculation (continued)

Since $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, *and* $i^{4k+3} = -i$, *for all* $k \in \mathbb{N} \cup \{0\}$, *we can divide the above series into four parts as follows.*

$$\begin{aligned} e^{yi} &= \sum_{k=0}^{\infty} \frac{y^{4k} i^{4k}}{(4k)!} + \sum_{k=0}^{\infty} \frac{y^{4k+1} i^{4k+1}}{(4k+1)!} + \sum_{k=0}^{\infty} \frac{y^{4k+2} i^{4k+2}}{(4k+2)!} + \sum_{k=0}^{\infty} \frac{y^{4k+3} i^{4k+3}}{(4k+3)!} \\ &= \sum_{k=0}^{\infty} \frac{y^{4k}}{(4k)!} + i \sum_{k=0}^{\infty} \frac{y^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{y^{4k+2}}{(4k+2)!} - i \sum_{k=0}^{\infty} \frac{y^{4k+3}}{(4k+3)!}. \end{aligned}$$

Combining the real parts and the imaginary parts together, it follows

$$\begin{aligned} e^{yi} &= \left(\sum_{k=0}^{\infty} \frac{y^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{y^{4k+2}}{(4k+2)!} \right) + i \left(\sum_{k=0}^{\infty} \frac{y^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{y^{4k+3}}{(4k+3)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \\ &= \cos y + i \sin y. \end{aligned}$$

Definition

For $z = x + yi$, $x, y \in \mathbb{R}$,

$$e^z = e^x(\cos y + i \sin y).$$

Proposition

For $z_1, z_2 \in \mathbb{C}$,

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

Remark

For $z \in \mathbb{C}$, *the complex exponential function also has the Taylor expansion*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

A complex number $z = x + yi$, $x, y \in \mathbb{R}$, can be identified as a point (x, y) in $\mathbb{R}^2$. We can interpret the algebraic manipulations of complex numbers in the following geometric way.

Addition

Given $z_1, z_2 \in \mathbb{C}$, we can construct a parallelogram with edges $\overline{0z_1}$ and $\overline{0z_2}$. Then the fourth vertex, different from 0, z_1 and z_2, corresponds to $z_1 + z_2$.

Subtraction

$z_1 - z_2$ denotes the vector starting from z_2 and ending at z_1.

Polar Coordinates

For $(x, y) \in \mathbb{R}^2$*, we have the polar coordinates*

$$(x, y) = (\rho\cos\theta, \rho\sin\theta),$$

where $\rho = \sqrt{x^2 + y^2}$, $\theta \in \mathbb{R}$*. The corresponding complex number* $z = x + yi$ *can be represented as*

$$z = x + yi = \rho\cos\theta + i\rho\sin\theta = \rho(\cos\theta + i\sin\theta).$$

By using Euler's formula, $\cos\theta + i\sin\theta = e^{i\theta}$*, we obtain*

$$z = \rho e^{i\theta}.$$

Definition

For $z = x + yi = \rho e^{i\theta} \in \mathbb{C}$, $\rho = \sqrt{x^2 + y^2}$ is called the modulus of z, denoted by $|z|$. That is, the modulus of z is

$$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}.$$

And for $z \neq 0$, we call θ an argument of z and define $\arg z$ to be the set of all argument of z.

Example

(i) $|-3 + 2i| = \sqrt{13}$.

(ii) $|1 + 4i| = \sqrt{17}$.

Remark

(i) *Geometrically,* $|z|$ *is the distance between* (x, y) *and the origin.*

(ii) $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$ *and* $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$.

(iii) *For* $z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1||z_2|$. *And* $\left|z^{-1}\right| = |z|^{-1}$ *if* $z \neq 0$.

(iv) $|z^n| = |z|^n$ *for* $z \in \mathbb{C}$, $n \in \mathbb{N}$.

(v) *For* $z = 0$, θ *is undefined.*

(vi) *For* $z \neq 0$, θ *is defined up to* $2k\pi$, $k \in \mathbb{Z}$. *If we restrict* θ *to be a number in* $(-\pi, \pi]$, *then the argument for a complex number can be uniquely determined. That is, there is a unique* $\Theta \in (-\pi, \pi]$ *such that* $\Theta \in \arg z$. *We call* Θ *the principal argument of* z, *denoted by* $\operatorname{Arg} z$.

(vii) *For* $z \neq 0$,

$$\arg z = \{\operatorname{Arg} z + 2k\pi : k \in \mathbb{Z}\}.$$

Example

$$\mathrm{Arg}(-1-i) = -\frac{3\pi}{4}.$$

$$\arg(-1-i) = \left\{-\frac{3\pi}{4} + 2k\pi : k \in \mathbb{Z}\right\}.$$

Proposition (Triangle inequality)

For $z_1, z_2 \in \mathbb{C}$*,*

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof.

For the second inequality, we can construct a triangle with vertices 0, z_1 and $z_1 + z_2$. Then length of the edge between 0 and $z_1 + z_2$ if bounded by the sum of the length of the other two. The inequality then follows. As for the first inequality, we can apply the inequality we just proved to get

$$|z_1| = |(z_1 + z_2) + (-z_2)| \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|.$$

That is,

$$|z_1| - |z_2| \leq |z_1 + z_2|.$$

Interchanging the roles of z_1 and z_2, we obtain

$$|z_2| - |z_1| \leq |z_1 + z_2|.$$

The last two inequalities complete the proof. □