27/01/2021 MATH 2230 B

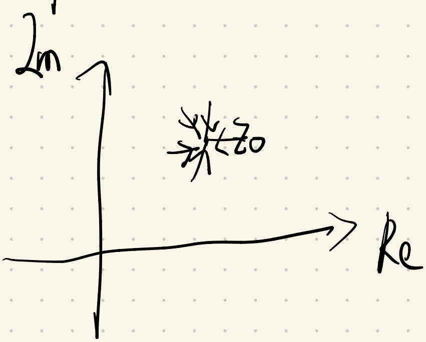
1. Derivative of $\mathbb{C}$-valued function.

Def: If f is differentiable at z_0

then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

& we call the limit derivative of f at z_0.



for $\forall$ direction approaching z_0, the limit should be the same.

Counter Example: f: $\mathrm{Re}\, z = x$.

$f' = 1$ from Re-axis

$f' = 0$ from Im-axis

Not differentiable.

2. Cauchy - Riemann Equation.

From necessary direction,

suppose f is differentiable at z_0,

then the limit must coincide when we approach z_0 from both real axis & imaginary axis.

From real axis:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(x_0 + \Delta x + iy_0) - f(x_0 + iy_0)}{\Delta x}$$

$$\begin{cases} z_0 = x_0 + iy_0 \\ f(z) = u(x,y) + iv(x,y) \end{cases} = \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \frac{iv(x_0 + \Delta x, y_0) - iv(x_0, y_0)}{\Delta x} \right]$$

$$= \boxed{u_x + iv_x}$$

From imaginary axis:

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{f(x_0 + i(y_0 + \Delta y)) - f(x_0 + iy_0)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\,\frac{v(x_0, y_0 + \Delta y) - i\,v(x_0, y_0 + \Delta y)}{i\Delta y} \right]$$

$$= v_y + \frac{u_x}{i} = \boxed{v_y - i u_x}$$

C-R equation

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If f does not satisfy C-R at z_0, then f is not differentiable at z_0.

Counter Example.

$$f = \begin{cases} \dfrac{x^3 y (y - ix)}{x^6 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$$\left| \frac{x^3 y (y - ix)}{x^6 + y^2} \right| \leq \frac{|x^3 y^2 + x^4 y|}{x^6 + y^2}$$

$$2|x|^3 y \leq x^6 + y^2$$

$$\leq \frac{|x|^3 y^2 + x^4 y}{2|x|^3 y} = \frac{y}{2} + \frac{|x|}{2}.$$

When $y, x \to 0$, $f \to 0$.

f is cts at 0.

$z = re^{i\theta}$, fixed θ, $\lim_{r \to 0} f \to 0$.

$$f = \frac{r^5 \cos^3\theta \sin\theta (\sin\theta - i\cos\theta)}{r^6 \cos^6\theta + r^2 \sin^2\theta}$$

$$\lim_{r \to 0} \frac{f}{z} = \lim_{r \to 0} r^2 \frac{\cos^3\theta \sin\theta}{r^4 \cos^6\theta + \sin^2\theta} \frac{\sin\theta - i\cos\theta}{\cos\theta + \sin\theta i}$$

$$= 0 \quad \text{for fixed } \theta.$$

$$\frac{f}{z} = r^2 \frac{\cos^3\theta \sin\theta}{r^4 \cos^6\theta + \sin^2\theta} \frac{\sin\theta - i\cos\theta}{\cos\theta + \sin\theta i}$$

$$\left|\frac{f}{z}\right| = r^2 \frac{\cos^3\theta \sin\theta}{r^4 \cos^6\theta + \sin^2\theta}$$

$$= r^2 \frac{1}{r^4 \frac{\cos^3\theta}{\sin\theta} + \frac{\sin\theta}{\cos^3\theta}}$$

$$a = \frac{\cos^3\theta}{\sin\theta}$$

For $\theta \in (0, \frac{\pi}{2}]$ cts, when $\theta \to 0$

$a \to \infty$, $\theta = \frac{\pi}{2}$, $a = 0$,

a increases as $\theta \to 0$.

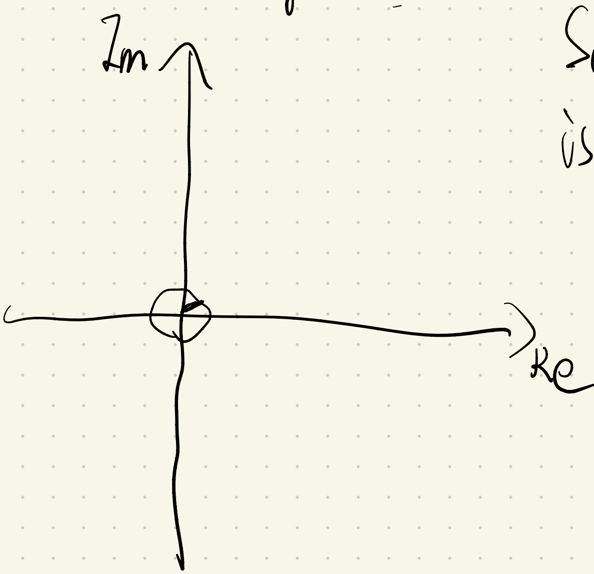
$$\left|\frac{f}{z}\right| = r^2 \frac{\cos^3\theta \sin\theta}{r^4\cos^6\theta + \sin^2\theta}$$

$$= r^2 \frac{1}{r^4 \frac{\cos^3\theta}{\sin\theta} + \frac{\sin\theta}{\cos^3\theta}}$$

$$= r^2 \frac{1}{r^4 a + \frac{1}{a}} \leq \frac{r^2}{2r^2} = \frac{1}{2}$$

$$r^4 a + \frac{1}{a} \geq 2r^2$$

$$a = \frac{1}{r^2}$$



So this convergence is not uniformly in θ. So f is not differentiable at 0.

3. Sufficiency.

Thm: For $f = u(x,y) + iv(x,y)$)

a) the 1st order derivative of u & v exist and cts.

($u, v \in C^1$).

b) $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$.

Then $f'(z_0)$ exists & $f'(z_0)$

$= u_x + iv_x$.

In conclusion, if $\mathbb{C}$-valued f has cts differentiable real & Imaginary part, C-R $\Leftrightarrow$ differentiability.

Proof: Omit.

4. Polar Coordinate.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{r}$$

$$= u_x \cos\theta + u_y \sin\theta.$$

$$\frac{\partial u}{\partial \theta} = -u_x r\sin\theta + u_y r\cos\theta$$

Remember $x = r\cos\theta$, $y = r\sin\theta$.

Similarly,

$$\begin{cases} v_r = v_x \cos\theta + v_y \sin\theta \\ v_\theta = -v_x r\sin\theta + v_y r\cos\theta. \end{cases}$$

Original C-R: $\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$

$$\begin{aligned} v_r &= -u_y\cos\theta + u_x \sin\theta, \\ v_\theta &= u_y r\sin\theta + u_x r\cos\theta \end{aligned} \Rightarrow \begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r. \end{cases}$$

Exp: $f = \frac{1}{z^2} = \frac{1}{r^2} e^{-i2\theta} = \frac{1}{r^2}(\cos 2\theta - i\sin 2\theta)$

$u = \frac{\cos 2\theta}{r^2}$, $v = -\frac{\sin 2\theta}{r^2} \in C^1(\mathbb{C}/\{0\})$

$$\begin{cases} r\frac{\partial u}{\partial r} = -\frac{2\cos 2\theta}{r^2} = v_\theta \\ u_\theta = -\frac{2\sin 2\theta}{r^2} = -r\frac{\partial v}{\partial r} \end{cases}$$

Differentiable on $\mathbb{C}/\{0\}$.

5. Another way on coordinate.

By z & $\bar{z}$, $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= (u_x + iv_x)\frac{1}{2} + (u_y + iv_y)\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2}u_x + i\frac{1}{2}v_x + \frac{i}{2}u_y - \frac{1}{2}v_y$$

$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)$$

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$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)$$

C-R: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ $= 0.$